I YEAR – I SEMESTER
COURSE CODE: 7MMA1C4

CORE COURSE-IV – ORDINARY DIFFERENTIAL EQUATIONS

Unit I

Linear equations with constant coefficients – Linear dependence and Independence – a formula for the Wronskian – non-homogenous equation – homogeneous equation of order n-initial value problems for n^{th} order equations – equations with real constants – non-homogeneous equations of order n.

Unit II

Linear equations with variable coefficients : Reduction of the order of a homogeneous equation – non-homogeneous equation-homogeneous equations with analytic coefficients – Legendre equation.

Unit III

Linear equations with regular singular points – Euler equations – second order equations with regular singular points – an example – second order equations with regular singular points – general case – exceptional cases – Bessel equation – Bessel equation (continued) – regular points at infinity.

Unit IV

Existence and uniqueness of solutions to first order equations : Equations with variables separated – exact equations – method of successive approximations – Lipchitz condition – convergence of the successive approximations.

Unit V

Nonlocal existence of solutions-approximations to solutions and uniqueness of solutions – Existence and uniqueness of solutions to systems and n^{th} order equations – existence and uniqueness of solutions to system.

Text Book

Earl A.Coddington, An Introduction to Ordinary Differential Equations – Prentice Hall of India, 1987.

Unit – I	Chapter - 2 sections 2.4 to 2.10
Unit – II	Chapter - 3 sections 3.5 to 3.8
Unit – III	Chapter - 4 sections 4.1 to 4.4 and 4.6 to 4.9
Unit – IV	Chapter - 5 sections 5.2 to 5.6
Unit – V	Chapter 5 & 6 sections 5.7 to 5.8 and 6.6

Books for Supplementary Reading and Reference:

1. D.Somasundaram, Ordinary Differential Equations, Narosa Publishing House, Chennai, 2002.
2. M.D.Raisinghania, Advanced Differential Equations, S.Chand and Company Ltd, New Delhi, 2001.

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Unit - I ODE

1

Linear equations with constant coefficients:

Definition Linear differential Equation

A Linear differential equation of order n with constant coefficients is an equation of the form

$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \dots + a_n y = b(x)$ where a_0 to $a_1, a_2, \dots, a_n$ are complex constants and b is some complex valued function on an interval I

Notation: $L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y$

Definition:

A Linear differential equation of order n $L(y) = b(x)$ is called a homogenous equation if $b(x) = 0$ for all x in I

A Linear differential equation of order n $L(y) = b(x)$ is called a non-homogenous equation if $b(x) \neq 0$ for all x in I

Note:

1. The second order homogenous equation is $L(y) = y'' + a_1 y' + a_2 y = 0$ where a_1 and a_2 are constants

2. The characteristic polynomial of $L(y) = 0$ is $p(r) = r^2 + a_1 r + a_2$

Theorem 1.1

Let a_1, a_2 be constants and consider the equation $L(y) = y'' + a_1 y' + a_2 y = 0$ if r_1, r_2 are distinct roots of the characteristic polynomial $p(r)$ then the function ϕ_1, ϕ_2 defined by $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are solution of $L(y) = 0$. If r_1 is a repeated root of $p(r)$ then the functions $\phi_1 \phi_2$ defined by $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = x e^{r_1 x}$ are the solutions of $L(y) = 0$

Linear dependence and Independence

Definition:

Two functions ϕ_1, ϕ_2 defined on an interval I are said to be linearly dependent on I. If there exist two constants c_1, c_2 not both zero such that $c_1 \phi_1(x) + c_2 \phi_2(x) = 0$ for all x in I

The functions ϕ_1, ϕ_2 are said to be linearly independent on I. if they are not dependent

Definition Wronskion imug NOV-2018

Wronskion of ϕ_1 and ϕ_2 is defined by

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$ 2

Theorem: 1.1

Two solution ϕ_1, ϕ_2 of $L(y)=0$ are linearly independent on an Interval I Iff $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I

linearly dependent $W(\phi_1, \phi_2)(x) = 0$

Proof: Suppose $W(\phi_1, \phi_2) \neq 0$ for all x in I

Let c_1, c_2 be constant such that

$c_1 \phi_1(x) + c_2 \phi_2(x) = 0$ for all x in I $\rightarrow$ 1

then $c_1 \phi_1'(x) + c_2 \phi_2'(x) = 0$ for all x in I $\rightarrow$ 2

For a fixed x equation 1 & 2 are Linearly homogenous equation satisfied by $c_1\ c_2$

Since the determinont of the coefficients, $W(\phi_1, \phi_2)(x)$ is not a zero

$c_1 = c_2 = 0$ is the only solution of equation 1 & 2 $\therefore$ ϕ_1 and ϕ_2 are linearly Independent of I

Conversely,

suppose $W(\phi_1,\phi_2)(x)=0$ for some x in I

Assume that ϕ_1, ϕ_2 are linearly Independent on I

Hence $W(\phi_1,\phi_2)(x_0)=0$

$$\left.\begin{aligned} \Rightarrow\ & c_1\phi_1(x_0) + c_2\phi_2(x_0) = 0 \\ \text{and}\ & c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = 0 \end{aligned}\right\} \rightarrow (3)$$

has a solutions c_1, c_2 where atleast one of those numbers is zero

Let c_1 and c_2 be such a solutions and consider the Function

$$\psi = c_1\phi_1 + c_2\phi_2$$

Now, $L(\psi) = L(c_1\phi_1 + c_2\phi_2) = 0$

$(3) \Rightarrow \psi(x_0) = 0$ and $\psi'(x_0) = 0$

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By uniqueness theorem

$$\psi(x) = 0$$

$$c_1\phi_1(x) + c_2\phi_2(x) = 0$$

which is contradiction to the fact that $\phi_1\ \phi_2$ are linearly independent on I

$\therefore\ W(\phi_1,\phi_2)(x) \neq 0$ for all x in I

Hence proved

Theorem: 2 Existence theorem

u.q.2m Nov 18 For any real x_0 and constants α, β there exists a solution ϕ of the initial value problem $L(y)=0$, $y(x_0)=\alpha$ and $y'(x_0)=\beta$ on $-\infty < x < \infty$

Theorem: 3 uniqueness theorem

A.P.-19 Let α, β be any two constants and Let x_0 be any real numbers. on any interval I containing x_0, there exist atmost one solution ϕ of the initial value problem $L(y)=0$, $y(x_0)=\alpha$ $y'(x_0)=\beta$

Theorem 1.2

Let ϕ_1, ϕ_2 be two solutions of $L(y)=0$ on an interval I and Let x_0 be any point in I. then ϕ_1, ϕ_2 are linearly independent on I. iff $W(\phi_1, \phi_2)(x_0) \neq 0$

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Proof: If ϕ_1, ϕ_2 are linearly independent on I. then $W(\phi_1, \phi_2) \neq 0$ for all x in I (By theorem 1.1)

In particular
$$W(\phi_1, \phi_2)(x_0) \neq 0$$

Conversely

Suppose $W(\phi_1, \phi_2)(x_0) \neq 0$ and c_1, c_2 are constants. such that $c_1\phi_1(x) + c_2\phi_2'(x) = 0$ for all x in I

then $c_1\phi_1(x_0) + c_2\phi_2(x_0) = 0$

and $c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = 0$

since $W(\phi_1, \phi_2)(x_0) \neq 0$

$c_1 = c_2 = 0$

Hence ϕ_1 and ϕ_2 are linearly independent on I.

Hence Proved

Theorem: 1.3

Let ϕ_1, ϕ_2 be any two linearly independent solutions of $L(y)=0$ on an Interval I. Every solution ϕ of $L(y)=0$ can be written uniquely as $\phi = c_1\phi_1 + c_2\phi_2$ where c_1, c_2 are constants

linear combination

Proof:

Let x_0 be a point on I

Since ϕ_1, ϕ_2 are linearly independent on I $W(\phi_1, \phi_2)(x_0) \neq 0$ (by theorem 1.2)

Let $\phi(x_0) = \alpha$ $\phi'(x_0) = \beta$ and consider the equation

$$\left.\begin{aligned} c_1 \phi_1(x_0) + c_2 \phi_2(x_0) &= \alpha \\ c_1 \phi_1'(x_0) + c_2 \phi_2'(x_0) &= \beta \end{aligned}\right\} \to (1)$$

For the constants c_1, c_2

since the determinant of the co-efficient of $c_1, c_2 \neq 0$, there is an unique pair of constant c_1, c_2 satisfying these equations

Choose c_1, c_2 to be these constants

then $\psi = c_1\phi_1 + c_2\phi_2$ is a function such that

$\psi(x_0) = \phi(x_0), \psi'(x_0) = \phi(x_0)$ and $L(\psi) = 0$

By uniqueness theorem

$$\psi = \phi \text{ on } I$$

(ie) $\phi = c_1\phi_1 + c_2\phi_2$

Hence proved

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Problem:-

(1) The functions ϕ_1, ϕ_2 are defined below exists for $-\infty < x < \infty$. Determined whether they are linearly independent or Dependent

1. $\phi_1(x) = x$, $\phi_2(x) = e^{rx}$, r is complex AP-19
2. $\phi_1(x) = \cos x$, $\phi_2(x) = \sin x$
3. $\phi_1(x) = x^2$, $\phi_2(x) = 5x^2$ Apr-19
4. $\phi_1(x) = \sin x$, $\phi_2(x) = e^{ix}$
5. $\phi_1(x) = \cos x$, $\phi_2(x) = 3(e^{ix} + e^{-ix})$

Solution

Given $\phi_1(x) = x$, $\phi_2(x) = e^{rx}$

$\phi_1'(x) = 1$, $\phi_2'(x) = re^{rx}$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} x & e^{rx} \\ 1 & re^{rx} \end{vmatrix}$$

$= rxe^{rx} - e^{rx}$

$= (rx-1)e^{rx}$

$\neq 0$

$\phi_1(x)$ and $\phi_2(x)$ are linearly independent

(ii) Given $\phi_1(x) = \cos x \quad \phi_2(x) = \sin x$

$\phi_1'(x) = -\sin x \quad \phi_2'(x) = \cos x$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

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$= \cos^2 x + \sin^2 x = 1$

$\neq 0$

$\phi_1(x)$ and $\phi_2(x)$ are linearly independent

(iii) Given $\phi_1(x) = x^2 \quad , \phi_2(x) = 5x^2$

$\phi_1'(x) = 2x \quad \phi_2'(x) = 10x$

$$W(\phi_1, \phi_2) = \begin{vmatrix} x^2 & 5x^2 \\ 2x & 10x \end{vmatrix}$$

$= 10x^3 - 10x^3$

$= 0$

$\phi_1(x)$ and $\phi_2(x)$ are linearly dependent

(iv) Given $\phi_1(x) = \sin x \quad \phi_2(x) = e^{ix}$

$\phi_1'(x) = \cos x \quad \phi_2(x) = \cos x + i\sin x$

$\phi_2'(x) = -\sin x + i\cos x$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \sin x & \cos x + i\sin x \\ \cos x & -\sin x + i\cos x \end{vmatrix}$$

$= -\sin^2 x + i\sin x\cos x - \cos^2 x - i\sin x\cos x$

$= -1 \neq 0$

$\phi_1(x)$ and $\phi_2(x)$ are linearly independent

(v) Given $\phi_1(x) = \cos x$ $\quad \phi_2(x) = 3(e^{ix} + e^{-ix})$

$\phi_1'(x) = -\sin x \quad \phi_2'(x) = -6\sin x$ $\quad 6\cos x$

$\cos x + i\sin x + \cos x - i\sin x$

$= 2\cos x$

$= 3(2\cos x)$

$= 6\cos x$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \cos x & 6\cos x \\ -\sin x & -6\sin x \end{vmatrix}$$

$= -6\sin x\cos x + 6\sin x\cos x$

$= 0$

$\phi_1(x)$ and $\phi_2(x)$ are linearly dependent

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Formula for the Wronskian

Theorem: 1.4

Vq If ϕ_1, ϕ_2 are two solutions of $L(y)=0$ on an interval I containing a point x_0 then $W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)}$ $W(\phi_1, \phi_2)(x_0)$

Proof:- Let $L(y) = y'' + a_1y' + a_2y = 0$

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Since ϕ_1, ϕ_2 are two solutions of $L(y)=0$

$L(\phi_1) = 0 \Rightarrow \phi_1'' + a_1\phi_1' + a_2\phi_1 = 0 \rightarrow (1)$

$L(\phi_2) = 0 \Rightarrow \phi_2'' + a_1\phi_2' + a_2\phi_2 = 0 \rightarrow (2)$

$(1) \times \phi_2 \Rightarrow \phi_1''\phi_2 + a_1\phi_1'\phi_2 + a_2\phi_1\phi_2 = 0$

$(2) \times \phi_1 \Rightarrow \phi_2''\phi_1 + a_1\phi_2'\phi_1 + a_2\phi_1\phi_2 = 0$

$\phi_1''\phi_2 - \phi_2''\phi_1 + a_1(\phi_2\phi_1' - \phi_2'\phi_1) = 0 \rightarrow (3)$

We know that,

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}$$

$W(\phi_1, \phi_2)(x) = \phi_1(x)\phi_2'(x) - \phi_1'(x)\phi_2(x)$

Let $W = W(\phi_1, \phi_2)(x)$

then $W = \phi_1\phi_2' - \phi_1'\phi_2$

$\Rightarrow W' = \phi_1\phi_2'' + \phi_1'\phi_2' - \phi_1'\phi_2' - \phi_1''\phi_2$

$\Rightarrow W' = \phi_1\phi_2'' - \phi_1''\phi_2$

$\therefore$ (3) implies that

$$-(\phi_1\phi_2'' - \phi_1''\phi_2) - a_1(\phi_1\phi_2' - \phi_1'\phi_2) = 0$$

$$\Rightarrow -W' - a_1 W = 0$$

$$\Rightarrow W' + a_1 W = 0$$

$$\Rightarrow W' = -a_1 W$$

$$\Rightarrow \frac{W'}{W} = -a_1 \quad \cdots (8)$$

Integrating,

$$\int \frac{W'}{W} dx = \int -a_1 dx$$

$\Rightarrow \log W = -a_1 x + c$ where c is constant

$$\Rightarrow W = e^{-a_1 x + c} = e^{-a_1 x} e^{c}$$

$\Rightarrow W = k\, e^{-a_1 x} \longrightarrow (4)$ where k is constant

At $x = x_0$,

$$W(\phi_1, \phi_2)(x_0) = k e^{-a_1 x_0}$$

$$\Rightarrow k = e^{a_1 x_0} W(\phi_1, \phi_2)(x_0)$$

Put the value of k in (4) we get

$$W = e^{a_1 x_0} W(\phi_1, \phi_2)(x_0)\, e^{-a_1 x}$$

$$W(\phi_1, \phi_2)(x) = e^{-a(x - x_0)} W(\phi_1, \phi_2)(x_0)$$

Hence proved

The non-homogenous equation of order 2 (The variations of constants)

Let $L(y) = y'' + a_1 y' + a_2 y = b(x)$

then the solution ψ of $L(y) = b(x)$ can be written as $\psi = \psi_p + c_1\phi_1 + c_2\phi_2$ where ϕ_1, ϕ_2 are two linear independent solutions of $L(y) = 0$ and ψ_p is a particular solution. Thus

$$\psi_p = u_1\phi_1 + u_2\phi_2$$

where u_1', u_2' satisfy the equation

$$\phi_1 u_1' + \phi_2 u_2' = 0 \rightarrow ①$$

$$\phi_1' u_1' + \phi_2' u_2' = b(x) \rightarrow ②$$

$$① \times \phi_2' \Rightarrow \phi_1 \phi_2' u_1' + \phi_2 \phi_2' u_2' = 0$$

$$② \times \phi_2 \Rightarrow \phi_1' \phi_2 u_1' + \phi_2 \phi_2' u_2' = \phi_2 b(x)$$

$$u_1'(\phi_1 \phi_2' - \phi_1' \phi_2) = -\phi_2 b(x)$$

$$\Rightarrow u_1' = \frac{-\phi_2 b(x)}{W(\phi_1, \phi_2)} \qquad W(\phi_1,\phi_2) = \phi_1\phi_2' - \phi_2\phi_1'$$

substitue the value of u_1' in ①

(9)

$$-\frac{\phi_1 \phi_2 b(x)}{W(\phi_1, \phi_2)} + \phi_2 u_2' = 0$$

$$\Rightarrow u_2' = \frac{\phi_1 b(x)}{W(\phi_1, \phi_2)}$$

$$\therefore u_1' = \frac{-\phi_2(x) b(x)}{W(\phi_1, \phi_2)(x)} \text{ and } u_2' = \frac{\phi_1(x) b(x)}{W(\phi_1, \phi_2)(x)}$$

Integrating,

$$u_1(x) = -\int_{x_0}^{x} \frac{\phi_2(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt \quad \text{and}$$

$$u_2(x) = \int_{x_0}^{x} \frac{\phi_1(t)\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt$$

$$\therefore \psi_p(x) = \int_{x_0}^{x} \frac{[\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)]\, b(t)}{W(\phi_1, \phi_2)(t)}\, dt$$

Problem:

① Solve: $y'' - y' - 2y = e^{-x}$

Solution

Let $L(y) = y'' - y' - 2y$.

The characteristic equation of $L(y) = 0$

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = 2, -1$$

The Two linearly independent solutions are (11)

$\phi_1(x) = e^{2x}$ and $\phi_2(x) = e^{-x}$

The particular integral is

$\psi_p = u_1\phi_1 + u_2\phi_2 \rightarrow (A)$

where u_1' and u_2' satisfies the equation

$\phi_1 u_1' + \phi_2 u_2' = 0$

$\phi_1' u_1' + \phi_2' u_2' = b(x)$

Here

$e^{2x} u_1' + e^{-x} u_2' = 0 \rightarrow (1)$

$2e^{2x} u_1' - e^{-x} u_2' = e^{-x} \rightarrow (2)$

$3e^{2x} u_1' = e^{-x}$

$u_1' = \frac{1}{3} e^{-3x}$

(10)

Substitute u_1' in (1) we get

$e^{2x} \frac{1}{3} e^{-3x} + e^{-x} u_2' = 0$

$\frac{1}{3} e^{-x} - e^{-x} u_2' = 0$

$u_2' = \frac{1}{3}$

$u_1 = \frac{1}{3}\int e^{-3x} dx$

$u_1(x) = \frac{-e^{-3x}}{9}$

$u_2(x) = -\frac{1}{3}\int dx = -x/3$

(A) becomes

$\psi p = -\frac{e^{-3x}}{9} e^{2x} - \frac{x}{3} e^{-x}$

$\psi p = \frac{-e^{-x}}{9} - \frac{x}{3} e^{-x}$

$\psi_\phi = \psi_p + c_1\phi_1 + c_2\phi_2$

Hence the required solution is

$\psi = c_1 e^{2x} + c_2 e^{-x} - \frac{e^{-x}}{9} - \frac{x e^{-x}}{3}$

$\psi = c_1 e^{2x} + c_2 e^{-x} - \frac{x e^{-x}}{3}$

2. Solve $y'' + 4y = \cos x$

Solution

Let $L(y) = y'' + 4y$

The characteristic equation of $L(y) = 0$

$r^2+4=0$

$r^2=-4$

$r=\pm 2i$

The two linearly independent solutions are

$\alpha=0$, $\beta=2$, $y(x)=e^{\alpha x}[A\cos\beta x+B\sin\beta x]$

$\phi_1(x)=\cos 2x$

$\phi_2(x)=\sin 2x$

The particular integral is

$$\phi_p=u_1\phi_1+u_2\phi_2 \longrightarrow (A)$$

Where u_1' and u_2' satisfies the equation

$$\phi_1 u_1'+\phi_2 u_2'=0$$

$$\phi_1' u_1'+\phi_2' u_2'=b(x)$$

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~~Where u_1' and u_2' satisfies the equation~~

~~$\phi_1 u_1'+\phi_2 u_2'=0$~~ Repeated

~~$\phi_1' u_1'+\phi_2' u_2'=b(x)$~~

Here

$$\cos 2x\, u_1'+\sin 2x\, u_2'=0 \longrightarrow (1)$$

$$-2\sin 2x\, u_1'+2\cos 2x\, u_2'=\cos x \longrightarrow (2)$$

$(1)\times 2\cos 2x \Rightarrow 2u_1'\cos^2 2x+2\cos 2x\sin 2x\, u_2'=0$

$(2)\times \sin 2x \Rightarrow -2u_1'\sin^2 2x+2\sin 2x\cos 2x\, u_2'=\sin 2x\cos x$

$\quad + \qquad - \qquad -$

$$2u_1'\cos^2 2x+2u_1'\sin^2 2x=-\sin 2x\cos x$$

$$2u_1'=-\sin 2x\cos x$$

$$u_1'=\frac{-\sin 2x\cos x}{2}$$

$$=\boxed{\frac{-\cos x\sin 2x}{2}}$$

Subtitue.

$$\cos 2x\left(\frac{-\sin 2x\cos x}{2}\right)+\sin 2x\, u_2'=0$$

$$\sin 2x\, u_2'=\frac{\sin 2x\cos x\cos 2x}{2}$$

$$\boxed{u_2'=\frac{\cos x\cos 2x\,\cancel{\sin 2x}}{2\,\cancel{\sin 2x}}}$$

Integrating

$$u_1 = -\frac{1}{2}\int \left\{\frac{1}{2}(\sin 3x - \sin x)\right\}dx$$

$$= -\frac{1}{4}\left\{\frac{-\cos 3x}{3} + \cos x\right\}$$

$$u_1 = +\frac{1}{4}\frac{\cos 3x}{3} \mp \frac{\cos x}{4}$$

$$\boxed{u_1 = \frac{\cos 3x}{12} \mp \frac{\cos x}{4}}$$

$$u_2 = \frac{1}{2}\int \frac{\cos 3x + \cos x}{2}\,dx \qquad \int \frac{1}{2}\times\frac{1}{2}\{\cos 3x + \cos x\}dx$$

$$= \frac{1}{4}\left\{\frac{\sin 3x}{3} + \sin x\right\}$$

$$= \frac{\sin 3x}{12} + \frac{\sin x}{4} \qquad \psi_p = u_1\phi_1 + u_2\phi_2$$

$$\psi_p = \left[\frac{\cos 3x}{12} - \frac{\cos x}{4}\right][\cos 2x] + \left[\frac{\sin 3x}{12} + \frac{\sin x}{4}\right][\sin 2x]$$

$$\psi_p = \frac{\cos 3x \cos 2x}{12} + \frac{\cos x \cos 2x}{4} + \frac{\sin 3x \sin 2x}{12} + \frac{\sin 2x \sin x}{4}$$

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$$= \frac{1}{12}\{\cos 3x \cos 2x + \sin 3x \sin 2x\} + \frac{1}{4}\{\cos x \cos 2x + \sin 2x \sin x\}$$

$$= \frac{1}{12}\{\cos x\} + \frac{1}{4}\cos x$$

$$= \frac{\cos x + 3\cos x}{12} = \frac{4\cos x}{12} = \frac{\cos x}{3}$$

$$\boxed{\psi_p = \frac{\cos x}{3}}$$

Hence the required solution is

$$\psi = A\cos 2x + B\sin 2x + \frac{1}{3}\cos x$$

Homogenous equation of order n :-

The homogenous equation of order n is

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

Characteristic polynomial of $L(y)=0$ is
$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_n$$

Example: Solve: $y''' - 3y' + 2y = 0$

Solution: Let $L(y) = y''' - 3y' + 2y = 0$

The characteristic equation of $L(y)=0$ is
$$r^3 - 3r + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & \downarrow & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

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$$(r-1)(r^2+r-2)=0$$
$$(r-1)(r+2)(r-1)=0$$

$r = 1, 1, -2$ are the roots

characteristic polynomial

Hence $\phi = (c_1 + c_2 x)e^{x} + c_3 e^{-2x}$

Where c_1, c_2 and c_3 are constants

which is the required of $L(y)=0$

Initial value problem for n^{th} order equation :-

Definition: The magnitude or length of ϕ by
$$\left[|\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2 \right]^{1/2}$$ and is denoted by $\|\phi(x)\|$

Theorem: 1.5 Apr-19

Let ϕ be any solution of $L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = 0$ on an interval I containing a point x_0 then for all x in I $\|\phi(x_0)\| e^{-k|x-x_0|} \le \|\phi(x)\| \le \|\phi(x_0)\| e^{k|x-x_0|}$ where $k = 1 + |a_1| + \cdots + |a_n|$

Proof: Let $u(x) = \|\phi(x)\|^2$

$$\Rightarrow u(x) = |\phi|^2 + |\phi'|^2 + \cdots + |\phi^{(n-1)}|^2$$

$$\Rightarrow u(x) = \phi\bar{\phi} + \phi'\bar{\phi}' + \cdots + \phi^{(n-1)}\overline{\phi^{(n-1)}}$$

$|z|^2 = z\bar{z}$

$|z| = |\bar{z}|$

$\Rightarrow u'(x) = \phi\bar{\phi}' + \bar{\phi}\phi' + \phi'\bar{\phi}'' + \phi''\bar{\phi}' + \dots + \phi^{(n-1)}\bar{\phi}^{(n)} + \phi^{(n)}\bar{\phi}^{(n-1)}$

$\Rightarrow |u'(x)| \le |\phi||\bar{\phi}'| + |\bar{\phi}||\phi'| + |\phi'||\bar{\phi}''| + |\phi''||\bar{\phi}'| + \dots + |\phi^{(n-1)}||\bar{\phi}^{(n)}| + |\phi^{(n)}||\bar{\phi}^{(n-1)}|$

$\Rightarrow |u'(x)| \le 2|\phi||\phi'| + 2|\phi'||\phi''| + \dots + 2|\phi^{(n-1)}||\phi^{(n)}| \longrightarrow (A)$

We know that

$(|b| - |c|)^2 \ge 0$ (14)

$|z| = \sqrt{x^2+y^2}$

$|\bar{z}| = \sqrt{x^2+y^2}$

$\Rightarrow |b|^2 + |c|^2 - 2|b||c| \ge 0$

$\Rightarrow 2|b||c| \le |b|^2 + |c|^2 \longrightarrow (1)$

Since ϕ is a solution of $L(y) = 0$

$L(\phi) = 0$

$\phi^{(n)} + a_1\phi^{(n-1)} + a_2\phi^{(n-2)} + \dots + a_n\phi = 0$

$\Rightarrow \phi^{(n)} = -(a_1\phi^{(n-1)} + a_2\phi^{(n-2)} + \dots + a_n\phi)$

$\Rightarrow |\phi^{(n)}| \le |a_1||\phi^{(n-1)}| + |a_2||\phi^{(n-2)}| + \dots + |a_n||\phi|$

(A) becomes

$|u'(x)| \le 2|\phi||\phi'| + 2|\phi'||\phi''| + \dots + 2|\phi^{(n-1)}| \{ |a_1||\phi^{(n-1)}| + |a_2||\phi^{(n-2)}| + \dots + |a_n||\phi| \}$

$|u'(x)| \le 2|\phi||\phi'| + 2|\phi'||\phi''| + \dots + 2|a_1||\phi^{(n-1)}|^2 + 2|a_2||\phi^{(n-1)}||\phi^{(n-2)}| + \dots + 2|a_n||\phi^{(n-1)}||\phi|$

Then by ①

$$|u'(x)| \le |\phi|^2+|\phi'|^2+|\phi'|^2+|\phi''|^2+\cdots+2|a_1|\,|\phi^{(n-1)}|^2$$
$$+|a_2|\,|\phi^{(n-1)}|^2+|a_2|\,|\phi^{(n-2)}|^2+\cdots+$$
$$|a_n|\,|\phi^{(n-1)}|^2+|a_n|\,|\phi|^2$$

$$\le |\phi|^2+2|\phi'|^2+2|\phi''|^2+\cdots+2|\phi^{(n-2)}|^2+|\phi^{(n-1)}|^2+$$
$$2|a_1|\,|\phi^{(n-1)}|^2+|a_2|\,|\phi^{(n-1)}|^2+|a_2|\,|\phi^{(n-2)}|^2$$
15 $$+\cdots+|a_n|\,|\phi^{(n-1)}|^2+|a_n|\,|\phi|^2$$

$$=(1+|a_n|)|\phi|^2+(2+|a_{n-1}|)|\phi'|^2+(2+|a_{n-2}|)$$
$$|\phi''|^2+\cdots+(2+|a_2|)\,|\phi^{(n-2)}|^2+(1+2|a_1|+|a_2|)+$$
$k \le 2k$
$$\cdots+(|a_n|)\,|\phi^{(n-1)}|^2$$

$$|u'(x)| \le 2k|\phi'|^2+2k|\phi'|^2+\cdots+2k|\phi^{(n-1)}|^2$$
$$\Rightarrow |u'(x)| \le 2k\{|\phi|^2+|\phi'|^2+\cdots+|\phi^{(n-1)}|^2\}$$
$$\Rightarrow |u'(x)| \le 2k\,u(x) \qquad k=1+|a_1|+\cdots+|a_n|$$
$$\Rightarrow -2ku(x) \le u'(x) \le 2ku(x) \longrightarrow ②$$

consider, $u'(x) \le 2ku(x)$
$$\Rightarrow u'(x)-2ku(x) \le 0$$
multiply by e^{-2kx}
$$\therefore\; e^{-2kx}u'(x)-2ke^{-2kx}u(x) \le 0$$
$$\Rightarrow \frac{d}{dx}\left[e^{-2kx}u(x)\right] \le 0$$

Integrating from x_0 to $x>x_0$

we get, $$\int_{x_0}^{x}\frac{d}{dx}\left[e^{-2kx}u(x)\right] \le 0$$
$$\left[e^{-2kx}u(x)\right]_{x_0}^{x} \le 0$$

$\Rightarrow e^{-2kx}u(x) - e^{-2kx_0}u(x_0) \le 0$

$\Rightarrow u(x) \le e^{2kx} e^{-2kx_0} u(x_0) \le 0$

$u(x) \le \frac{e^{-2kx_0}u(x_0)}{e^{-2kx}} \Rightarrow u(x) \le e^{2k(x-x_0)} u(x_0)$

$\Rightarrow \{u(x)\}^{1/2} \le \{e^{2k(x-x_0)1/2}\} \{u(x_0)\}^{1/2}$

$\Rightarrow ||\phi(x)|| \le e^{k(x-x_0)} ||\phi(x_0)||$ if $x > x_0 \rightarrow (3)$

Similarly, the left side of the inequality (2) becomes,

$e^{-k(x-x_0)} ||\phi(x_0)|| \le ||\phi(x)|| \le ...$

$$e^{-k(x-x_0)} ||\phi(x_0)|| \le ||\phi(x)|| \rightarrow (4)$$

if $x > x_0$

(3) and (4) gives

$$e^{-k(x-x_0)} ||\phi(x_0)|| \le ||\phi(x)|| \le e^{k(x-x_0)} ||\phi(x_0)||$$

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Similarly,

$$e^{-k(x_0-x)} ||\phi(x_0)|| \le ||\phi(x)|| \le e^{k(x_0-x)} ||\phi(x_0)||$$

if $x < x_0$

Hence

$$||\phi(x_0)|| e^{-k|x-x_0|} \le ||\phi(x)|| \le ||\phi(x_0)|| e^{k|x-x_0|}$$

Hence Proved

Theorem:- 1.6

Uniqueness Theorem

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any n constants and let x_0 be any real number on any interval I containing x_0 there exists atmost one solution ϕ of $L(y)=0$ satisfying $\phi(x_0)=\alpha_1$, $\phi'(x_0)=\alpha_2 \dots \dots$ $\phi^{(n-1)}(x_0)=\alpha_n$

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Proof:- Let ϕ and ψ be two solution of $L(y)=0$ satisfying the conditions $\phi(x_0)=\alpha_1$, $\phi'(x_0)=\alpha_2$ $\dots$ $\phi^{(n-1)}(x_0)=\alpha_n$

$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \ldots, \psi^{(n-1)}(x_0) = \alpha_n$

consider $\chi = \phi - \psi$ for all x in I

Now, $L(\chi) = L(\phi) - L(\psi) = 0$

$\therefore L(\chi) = 0$

and $\chi(x_0) = \phi(x_0) - \psi(x_0) = \alpha_1 - \alpha_1 = 0$

$\chi'(x_0) = \phi'(x_0) - \psi'(x_0) = \alpha_2 - \alpha_2 = 0$

$\vdots$

$\chi^{(n-1)}(x_0) = \phi^{(n-1)}(x_0) - \psi^{(n-1)}(x_0) = \alpha_n - \alpha_n = 0$

$\therefore ||\chi(x_0)||^2 = |\chi(x_0)|^2 + |\chi'(x_0)|^2 + \cdots + |\chi^{(n-1)}(x_0)|^2$

$||\chi(x_0)||^2 = 0$

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$\Rightarrow ||\chi(x_0)|| = 0$

By theorem : 1.5

$e^{-k|x-x_0|} ||\chi(x_0)|| \leq ||\chi(x)|| \leq ||\chi(x_0)|| e^{k|x-x_0|}$

$\Rightarrow 0 \leq ||\chi(x)|| \leq 0$

$\Rightarrow ||\chi(x)|| = 0$ for all x in I

$\Rightarrow \chi(x) = 0$, for all x in I

$\therefore \phi(x) = \psi(x)$; for all x in I

Definition:

The wronskian $W(\phi_1, \phi_2, \cdots \phi_n)$ of n functions $\phi_1, \phi_2, \cdots \phi_n$ having $n-1$ derivatives on an interval I is defined by

$$W(\phi_1, \phi_2, \cdots \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}$$

and its value at x is denoted by $W(\phi_1, \phi_2, \cdots \phi_n)(x)$

Theorem : 1.4

(Existence theorem)

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any constants and let x_0 be any real number there exists a solution ϕ of $L(y)=0$ on $-\infty < x < \infty$ satisfying $\phi(x_0)=\alpha_1$, $\phi'(x_0)=\alpha_2, \dots, \phi^{(n-1)}(x_0)=\alpha_n$

Proof :-

Let $\phi_1, \phi_2, \dots, \phi_n$ be any set of n linearly independent solutions of $L(y)=0$ on $-\infty < x < \infty$

there exists a unique constants $c_1, c_2, \dots c_n$ such that $\phi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n \rightarrow (1)$

is the solution of $L(y)=0$ satisfying given condition

$$\left.\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \dots + c_n\phi_n(x_0) &= \alpha_1 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \dots + c_n\phi_n'(x_0) &= \alpha_2 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \dots + c_n\phi_n^{(n-1)}(x_0) &= \alpha_n \end{aligned}\right\} \rightarrow (2)$$

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which is a system of n linear equation for $c_1, c_2, \dots, c_n$

then the determinante of its coefficient is $W(\phi_1, \phi_2, \dots, \phi_n) \neq 0$ since $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent

$\therefore$ There is a unique set of constants $c_1, c_2, \dots, c_n$ satisfying (2)

Hence $\phi = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$ is a desired solution.

Hence proved

(9)

Result: $W(\phi_1, \phi_2, \dots \phi_n)(x) = e^{-a_1(x-x_0)}$ Theorem of $W(\phi_1, \phi_2, \dots \phi_n)(x_0)$

① Find $W(\phi_1, \phi_2, \dots, \phi_n)(x)$ of $L(y) = y''' - 3r_1 y'' + 3r_1^2 y' - r_1^3 y$

solution: by using $W(\phi_1, \phi_2, \dots \phi_n)(x) = e^{-a(x-x_0)} W(\phi_1, \phi_2, \dots \phi_n)(x_0)$

Solution:

Let $L(y) = y''' - 3r_1 y'' + 3r_1^2 y' - r_1^3 y = 0$

The characteristic equation of $L(y) = 0$ is

$$m^3 - 3r_1 m^2 + 3r_1^2 m - r_1^3 = 0$$

$$\Rightarrow (m - r_1)^3 = 0$$

$$\Rightarrow m = r_1, r_1, r_1$$

Hence the linear independent solutions ϕ_1, ϕ_2, ϕ_3 of $L(y) = 0$ are

$$\phi_1(x) = e^{r_1 x}, \phi_2(x) = x e^{r_1 x} \quad \phi_3(x) = x^2 e^{r_1 x}$$

Now,

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix}$$

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$$= \begin{vmatrix} e^{r_1 x} & x e^{r_1 x} & x^2 e^{r_1 x} \\ r_1 e^{r_1 x} & r_1 x e^{r_1 x} + e^{r_1 x} & r_1 x^2 e^{r_1 x} + 2x e^{r_1 x} \\ r_1^2 e^{r_1 x} & r_1^2 x e^{r_1 x} + 2r_1 e^{r_1 x} & r_1^2 x^2 e^{r_1 x} + 4r_1 e^{r_1 x} + 2e^{r_1 x} \end{vmatrix}$$

At $x_0 = 0$

$$W(\phi_1, \phi_2, \phi_3)(x_0) = \begin{vmatrix} 1 & 0 & 0 \\ r_1 & 1 & 0 \\ r_1^2 & 2r_1 & 2 \end{vmatrix}$$

$$= 1(2) = 2$$

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-a(x-x_0)} W(\phi_1, \phi_2, \phi_3)(x_0)$$

$$\therefore W(\phi_1, \phi_2, \phi_3)(x) = 2e^{3r_1 x}$$

Equations with Real constants

① Find the solution of $y^{(4)} + y = 0$

Solution:- Let $L(y) = y^{(4)} + y = 0$

The characteristic equation of $L(y) = 0$ is

$$r^4 + 1 = 0$$

$\Rightarrow r^4 = -1$

$\Rightarrow r = (-1)^{1/4}$

$\Rightarrow r = (\cos(2n+1)\pi + i\sin(2n+1)\pi)^{1/4}$

$r = \cos\frac{(2n+1)}{4}\pi + i\sin\frac{(2n+1)}{4}\pi$

$n=0$

$r = \cos\frac{\pi}{4} + i\sin\pi/4$

$r = 1/\sqrt{2} + i\,1/\sqrt{2}$

$n=1$

$r = \cos\frac{3\pi}{4} + i\sin 3\pi/4$

$r = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$n=2$

$r = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$= -1/\sqrt{2} - i\,1/\sqrt{2}$

$n=3$ $\quad r = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

Hence the required solution is 20

$$y(x) = e^{1/\sqrt{2}\,x}\left(A\cos\frac{1}{\sqrt{2}}x + B\sin\frac{1}{\sqrt{2}}x\right) + e^{-\frac{1}{\sqrt{2}}x}\left(C\cos\frac{1}{\sqrt{2}}x + D\sin 1/\sqrt{2}\,x\right)$$

2. $y'' + y = 0$

Solution:- Nov-17

Let $L(y) = y'' + y = 0$

The characteristic equation of $L(y)=0$ is

$r^2 + 1 = 0$

$r = -1$; $r = \pm i$

Hence the required solution is

$y(x) = (A\cos x + B\sin x)$

3) $y'' - y = 0$

Solution:- Let $L(y) = y'' - y = 0$

The characteristic equation of $L(y)=0$ is

$r^2 - 1 = 0$

Ap-18. $r^2 = 1$

$r = \pm 1$

Hence the required solution is

$y(x) = Ae^{x} + Be^{-x}$

(4) $y^{(4)} - y = 0$

Solution: Let $L(y) = y^{(4)} - y = 0$

The characteristic equation $L(y) = 0$ is

$r^4 - 1 = 0$

$r^4 = 1$

$r = (1)^{1/4}$

$r = (\cos 2n\pi + i\sin 2n\pi)^{1/4}$

$= \cos \frac{2n\pi}{4} + i\sin \frac{2n\pi}{4}$

$n=0$ $r = \cos(0) + i\sin(0)$

$r = 1$

$n=1$ $r = \cos \pi/2 + i\sin \pi/2$

$= i$

$n=2$ $r = \cos \pi + i\sin \pi$ $\quad r = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{2}$

$= -1$

$n=3$ $r = \cos \frac{3\pi}{2} + i\sin\frac{3\pi}{2}$ 21

$= -i$

Hence the required solution

$y(x) = Ae^{x} + Be^{-x} + C\cos x + D\sin x$

A non-homogeneous Equation of order n

Let $L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x)$

$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$

where $\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$

* $U_k(x) = \int_{x_0}^{x} \frac{W_k(t)\, b(t)}{W(\phi_1, \phi_2, \cdots \phi_n)(t)}\, dt$

* $\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x_0} \frac{W_k(t)\, b(t)}{W(\phi_1 \phi_2 \phi_3 \cdots \phi_n)(t)}\, dt$

where W_k is the determinant obtained from $W(\phi_1, \phi_2 \cdots \phi_n)$ by replacing k^{th} column by $0, 0, \ldots 0, 1$

Imp
1. Solve $y''' + y'' + y' + y = 1$ which satisfies $\psi(0) = 0$ $\psi'(0) = 1$ $\psi''(0) = 0$

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solution Let $L(y) = y''' + y'' + y' + y = 1$

The characteristic equation of $L(y) = 0$ is

$r^3+r^2+r+1=0$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & 1 & 1 \\ & 0 & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}$$

$(r+1)(r^2+1)=0$

$r=-1,\ r^2=-1 \Rightarrow r=\pm i$

The linear indenpendent solution are,

$\phi_1(x)=e^{-x}$, $\phi_2(x)=\cos x$, $\phi_3(x)=\sin x$

Now, $\psi_p = u_1\phi_1+u_2\phi_2+u_3\phi_3$

$$W(\phi_1,\phi_2,\phi_3)(x)=\begin{vmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix}$$

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$$=\begin{vmatrix} e^{-x} & \cos x & \sin x \\ -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \end{vmatrix}$$

$= e^{-x}(\sin^2 x+\cos^2 x)-\cos x(e^{-x}\sin x-e^{-x}\cos x)+$
$\sin x(e^{-x}\cos x+e^{-x}\sin x)$

$=e^{-x}(1)+e^{-x}(\cos^2 x+\sin^2 x)$

$=2e^{-x}$

$$W_1=\begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix}=1.$$

$-\cos x(\cos x)+\sin x(-\sin x)$

$\cos^2 x+\sin^2 x$

$$W_2=\begin{vmatrix} e^{-x} & 0 & \sin x \\ -e^{-x} & 0 & \cos x \\ e^{-x} & 1 & -\sin x \end{vmatrix}$$

$=-1(e^{-x}\cos x+e^{-x}\sin x)$

$=-e^{-x}(\cos x+\sin x)$

$$W_3=\begin{vmatrix} e^{-x} & \cos x & 0 \\ -e^{-x} & -\sin x & 0 \\ e^{-x} & \cos x & 1 \end{vmatrix}$$

$e^{-x}(\cos x)+\sin x(-e^{-x}$

$e^{-x}(\cos x-\sin x)$

$=-e^{-x}(\sin x-\cos x)$

$=e^{-x}(-\sin x)-\cos x(-e^{-x})$

$=e^{-x}(-\sin x+\cos x)$

$= -e^{-x}\sin x + e^{-x}\cos x$

$= e^{-x}(\cos x - \sin x)$

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt$$

let $x_0 = 0$

$$u_1(x) = \int_0^x \frac{W_1(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt$$

$$= \int_0^x \frac{1}{2e^{-t}}\,dt$$

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$$= \frac{1}{2}\int_0^x e^t\,dt \quad = \frac{1}{2}\left[e^t\right]_0^x$$

$\sin x$
$\cos x$

$\cos x$
$-\sin x$

$$u_1(x) = \frac{1}{2}(e^x - 1)$$

$$u_2(x) = \int_0^x \frac{W_2(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt$$

$$= \int_0^x \frac{-e^{-t}(\cos t + \sin t)\,dt}{2e^{-t}}$$

$[-\sin t, -\cos t]$

$$= \frac{-1}{2}\left[\sin t - \cos t\right]_0^x$$

$$u_2(x) = -\frac{1}{2}\left[\sin x - \cos x + 1\right]$$

$$u_3(x) = \int_0^x \frac{W_3(t)\,dt}{W(\phi_1,\phi_2,\phi_3)(t)}\,dt$$

$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)\,b(t)}{W(\phi_1,\phi_2,\phi_3)}$

$$= \int_0^x \frac{e^{-t}(\cos t - \sin t)}{2e^{-t}}\,dt$$

$$= \frac{1}{2}\left[\sin t + \cos t\right]_0^x$$

$$u_3(x) = \tfrac{1}{2}(\sin x + \cos x - 1)$$

$$u_1(x) = \tfrac{1}{2}(e^x - 1) \Rightarrow u_1(x) = \tfrac{1}{2}e^x$$

$$u_2(x) = -\tfrac{1}{2}(\sin x - \cos x + 1) \Rightarrow u_2(x) = -\tfrac{1}{2}(\sin x - \cos x)$$

$$u_3(x) = \frac{1}{2}(\sin x + \cos x - 1) \Rightarrow u_3(x) = \tfrac{1}{2}(\sin x + \cos x)$$

$$\psi_p = u_1\phi_1 + u_2\phi_2 + u_3\phi_3$$

$= \frac{1}{2} e^{x} e^{-x} - \frac{1}{2}(\sin x - \cos x)\cos x + \frac{1}{2}(\sin x + \cos x)\sin x$

$= \frac{1}{2} - \frac{1}{2}\sin x\cos x + \frac{1}{2}\cos^2 x + \frac{1}{2}\sin^2 x + \frac{1}{2}\sin x\cos x$

$\boxed{\psi_p = 1.}$ $\quad \frac{1}{2} + \frac{1}{2} = 1$

$\psi(x) = \psi_p + c_1\phi_1 + c_2\phi_2 + c_3\phi_3$

$\psi(x) = c_1 e^{-x} + c_2\cos x + c_3\sin x + 1 \rightarrow (A)$

$\psi'(x) = -c_1 e^{-x} - c_2\sin x + c_3\cos x$

$\psi''(x) = c_1 e^{-x} - c_2\cos x - c_3\sin x$

$\psi(0) = 0 \Rightarrow c_1 + c_2 + 1 = 0 \rightarrow (1)$

$\psi'(0) = 1 \Rightarrow -c_1 + c_3 = 1 \rightarrow (2)$

$\psi''(0) = 0 \Rightarrow c_1 - c_2 = 0 \rightarrow (3)$

$(1) + (3)$ $\quad 2c_1 + 1 = 0 \Rightarrow c_1 = -\frac{1}{2}$

$(3) \Rightarrow -\frac{1}{2} - c_2 = 0$

$c_2 = -\frac{1}{2}$

$(2) \Rightarrow -(-\frac{1}{2}) + c_3 = 1$

$c_3 = \frac{1}{2}$

$c_1 = -\frac{1}{2}$

$c_2 = -\frac{1}{2}$

$c_3 = \frac{1}{2}$

(A) becomes

$\psi(x) = -\frac{1}{2} e^{-x} - \frac{1}{2}\cos x + \frac{1}{2}\sin x + 1$

Hence the required solution is

$\psi(x) = 1 - \frac{1}{2}(e^{-x} + \cos x - \sin x)$

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Unit - II

Linear differential equation with variable coefficient :-

(1)

Reduction of the order of the homogeneous equation:-

Let $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \cdots + a_n(x) y$

Problem:-

UQ Apr-18

1. Find the second independent solution of $y'' - \frac{2}{x^2} y = 0$ with the solution $\phi_1(x) = x^2$, $0 < x < \infty$

solution

Let $L(y) = y'' - \frac{2}{x^2} y = 0$

Let an another independent solution is of the form

$$\phi_2(x) = u\,\phi_1(x)$$
$$= ux^2$$
$$\phi_2'(x) = u2x + x^2u'$$
Again diff
$$\phi_2''(x) = 2u + 2xu' + x^2u'' + u'2x$$
$$\phi_2''(x) = x^2u'' + 4xu' + 2u.$$
Since $\phi_2(x)$ is a solution of $L(y)=0$
$$L(\phi_2)=0$$
$$\Rightarrow \phi_2''(x) - \frac{2}{x^2}\phi_2(x) = 0$$
$$\Rightarrow x^2u'' + 4xu' + 2u - 2/x^2\,(ux^2) = 0.$$
$$\Rightarrow x^2u'' + 4xu' = 0 \rightarrow ①$$
Let $u' = v \Rightarrow u'' = v'$

$$① \Rightarrow x^2v' + 4xv = 0 \qquad \Rightarrow x(xv' + 4v) = 0$$
$$\Rightarrow xv' + 4v = 0$$
$$xv' = -4v$$
$$\Rightarrow v'/v = -4/x$$
Integrating
$$\log v = -4\log x + \log c$$
$$= \log x^{-4} + \log c$$
$$\log v = \log cx^{-4}$$
$$v = cx^{-4}$$
$$\therefore u' = cx^{-4}$$
$$\Rightarrow u(x) = c\int x^{-4}dx = \frac{x^{-3}}{-3} \text{ constant}$$
$$\Rightarrow u(x) = x^{-3}.$$
Hence $\phi_2(x) = x^{-3}x^2 = x^{-1} = 1/x$

$$\phi_2(x) = 1/x$$

I. Verify that the function ϕ_1 satisfies the equation and find the second independent solution

(i) $x^2y'' - 7xy' + 15y = 0 \qquad \phi_1(x) = x^3 \quad (x>0)$

(ii) $y'' - 4xy' + (4x^2-2)y = 0 \qquad \phi_1(x) = e^{x/2}$

(iii) $xy'' - (x+1)y' + y = 0 \qquad \phi_1(x) = e^x$

Solution:- (i) Let $L(y) = x^2y'' - 7xy' + 15y \qquad \phi_1(x) = x^3$

$$L(\phi_1) = x^2\phi_1'' - 7x\phi_1' + 15\phi_1$$

$$= 6x^3 - 21x^3 + 15x^3$$

$$-15x^3 + 15x^3$$

$L(\phi_1) = 0$

Hence ϕ is the solution of $L(y)=0$. Another solution of $L(y)=0$ is of the form

$$\phi_2(x) = u\phi_1(x)$$

$$= ux^3$$

$$\phi_2'(x) = u\,3x^2 + x^3u'$$

$$\phi_2''(x) = u6x + 3x^2u' + x^3u'' + u'3x^2$$

$$= x^3u'' + 6x^2u' + 6xu$$

since $\phi_2(x)$ is a solution of $L(y)=0$

$$L(\phi_2) = 0$$

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$$\Rightarrow x^2\phi_2'' - 7x\phi_2' + 15\phi_2 = 0$$

$$\Rightarrow x^2(x^3u'' + 6x^2u' + 6xu) - 7x(3x^2u + x^3u') + 15ux^3 = 0$$

$$\Rightarrow x^5u'' + 6x^4u' + 6x^3u - 21x^3u - 7x^4u' + 15ux^3 = 0$$

$$\Rightarrow x^5u'' - x^4u' = 0 \quad \div x^4$$

$$\Rightarrow xu'' - u' = 0$$

Let $v = u' \Rightarrow v' = u''$

$$\Rightarrow xv' - v = 0$$

$$\Rightarrow xv' = v$$

$$v'/v = x^{-1}$$

Integrating

$$\log v = \log x + \log c$$

$$v = xc$$

$$u' = xc$$

$$u(x) = c\int x\,dx$$

Integrating

$$u(x) = c\,x^2/2$$

$$\phi_2(x) = x^2 \cdot x^3$$

$$\phi_2(x) = x^5$$

2M

(ii) Let $L(y) = y'' - 4xy' + (4x^2 - 2)y = 0$

$$\phi_1(x) = e^{x^2}$$

Let $L(\phi_1) = \phi_1'' - 4x\phi_1' + (4x^2 - 2)\phi_1$

$= 2e^{x^2} + 4x^2e^{x^2} - 8x^2 \cdot e^{x^2} + 4x^2e^{x^2} - 2e^{x^2}$

$L(\phi_1) = 0$

Hence ϕ is the solution of $L(y)=0$. Another solution of $L(y)=0$ is of the form

$$\phi_2(x) = u\phi_1(x)$$

$$= ue^{x^2}$$

$$\phi_2'(x) = (u2e^{x^2}x) + e^{x^2}u'$$

$$\phi_2''(x) = u(2e^{x^2} + 4x^2e^{x^2}) + 2xe^{x^2}.u' + e^{x^2}.u'' + u'.e^{x^2}2x$$

$$\phi_2''(x) = e^{x^2}u'' + 4xe^{x^2}.u' + (2e^{x^2} + 4x^2e^{x^2})u$$

since ϕ_2 is a solution of $L(y)=0$

$$L(\phi_2) = 0 \quad (4)$$

$$\phi_2'' - 4x\phi_2' + (4x-2)\phi_2 = 0$$

$$e^{x^2}.u'' + 4xe^{x^2}.u' + (2e^{x^2} + 4x^2e^{x^2})u - 8x^2e^{x^2}u - 4xe^{x^2}u' + 4x^2ue^{x^2} - 2ue^{x} = 0$$

$$e^{x^2}.u'' = 0$$

$$u'' = 0$$

Let $u' = v$, $u'' = v'$

$(1) \Rightarrow v' = 0$

Integrate $v = c$

$u' = c$

Integrate $u = cx$

$\Rightarrow u(x) = x$

Hence $\phi_2(x) = u\,\phi_1(x)$

$= x.e^{x^2}$

type - II

Problem :- 1

NOV-18

One Solution of $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$ for $x>0$ is $\phi_1(x) = x$. Find a basis for the solutions for $x>0$

solution:-

Let $L(y) = x^3y''' - 3x^2y'' + 6xy' - 6y$

Given $\phi_1(x) = x$ is the one solution of $L(y)=0$. Another solution of $L(y)=0$ is of the form $u\phi_1(x)$

$$\text{Let } \phi_2(x) = u\phi_1(x) = ux$$

$$\phi_2'(x) = u + xu'$$

$\phi_2''(x) = u' + xu'' + u'$

$= xu'' + 2u'$

$\phi_2'''(x) = xu''' + u'' + 2u''$

$= xu''' + 3u''$

Since ϕ_2 is the solution of $L(y)=0$

$L(\phi_2)=0$ (5)

$\Rightarrow x^3\phi_2''' - 3x^2\phi_2'' + 6x\phi_2' - 6\phi_2 = 0$

$\Rightarrow x^4u_2''' + 3x^3u'' - 3x^3u'' - 6x^2u' + \cancel{6xu} + 6x^2u' - \cancel{6ux} = 0$

$\Rightarrow \quad x^4u_2''' = 0$

$\Rightarrow u''' = 0.$

Integrating $\quad u'' = c$ (or) $u'' = 0$

Integrating $\quad u' = cx$ or $u' = c$ or $u' = 0$

Integrate $\quad u = cx^2/2$ or $u = cx$ or $u = c$

Since the solution of $L(y)=0$ is of the form $u\,\phi_1(x)$

The independent solutions are $c/_2\, x^3, cx^2$ and cx

$\therefore$ The basis for the solutions of $L(y)=0$ are x, x^2, and x^3

Problem :- 2

Two solutions of $x^3y''' - 3xy' + 3y = 0$ $(x>0)$ are $\phi_1(x) = x$, $\phi_2(x) = x^3$. Find a third independent solution

Solution:

Let $L(y) = x^3y''' - 3xy' + 3y = 0$

Given $\phi_1(x) = x$ $\phi_2(x) = x^3$

Another solution of $L(y)=0$ is of the form $u\,\phi_1(x)$

Let $\phi_3(x) = u\,\phi_1(x) = ux$

$\phi_3'(x) = u + xu'$

$\phi_3''(x) = u' + xu'' + u'$

$= xu'' + 2u'$

$\phi_3'''(x) = xu''' + u'' + 2u''$

$= xu''' + 3u''$

Since ϕ_3 is the solution of $L(y)=0$

$$L(\phi_3)=0$$

$\Rightarrow x^3 \phi_3'''(x) - 3x\phi_3' + 3\phi_3 = 0$

$\Rightarrow x^3(xu''' + 3u'') - 3x(u + xu') + 3(ux) = 0$

$\Rightarrow x^4u''' + 3x^3u'' - 3xu - 3x^2u' + 3ux = 0$

$\Rightarrow x^4u''' + 3x^3u'' - 3x^2u' = 0 \quad \div x^2$

$\Rightarrow x^2u''' + 3xu'' - 3u' = 0 \longrightarrow (1)$

let $v = u' \Rightarrow v' = u''$

$\Rightarrow v'' = u'''$

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$\therefore (1) \Rightarrow x^2v'' + 3xv' - 3v = 0$

let $I(v) = x^2v'' + 3xv' - 3v$ one of the solution of

$I(v)=0$ is $= \left(\frac{\phi_2}{\phi_1}\right)' = \left(\frac{x^3}{x}\right)' = (x^2)'$

This Implies that, $= 2x$

$\varphi_1(x) = x$ is one of the solution of $I(v)=0$

Hence another solution of $I(v)=0$ is of the form $w\varphi_1(x)$

let $\varphi_2(x) = w\varphi_1(x) = wx$

$\varphi_2'(x) = w + xw'$

$\varphi_2''(x) = w' + xw'' + w'$

$= xw'' + 2w'$

Since $\varphi_2(x)$ is a solution of $I(v)=0$

$$I(\varphi_2)=0$$

$\Rightarrow x^2\varphi_2'' + 3x\varphi_2' - 3\varphi_2 = 0$

$\Rightarrow x^2(xw'' + 2w') + 3x(w + xw') - 3(wx) = 0$

$\Rightarrow x^3w'' + 2x^2w' + \cancel{3xw} + 3x^2w - \cancel{3wx} = 0$

$\Rightarrow x^3w'' + 5x^2w' = 0 \quad \div x^2$

$xw'' + 5w' = 0 \rightarrow (2)$

Let $v_1 = w' \Rightarrow v_1' = w''$

$(2) \Rightarrow xv_1' + 5v_1 = 0$

$xv_1' = -5v_1$

$\frac{v_1'}{v_1} = -5/x$

Integrate. $\log v_1 = -5\log x + \log c$

$\log v_1 = \log c\, x^{-5}$

$v_1 = c\, x^{-5}$

ie) $w' = c x^{-5}$

Integrate

$w = c\int x^{-5} dx$

$w = \frac{x^{-4}}{-4}$

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$\therefore \psi_2(x) = w x \Rightarrow \psi_2(x) = x^{-4} x$

$\therefore \psi_2(x) = x^{-3}$

$\therefore x$ and x^{-3} are the basis for $I(V)=0$

If $v = x$	If $v = x^{-3}$
$\Rightarrow u' = x$	$u' = x^{-3}$
$\Rightarrow u = x^2/2$	$u = \frac{x^{-2}}{2}$
$u_1 = x^2/2$	$u_2 = \frac{x^{-2}}{-2}$

$\therefore$ The solution of $L(y)=0$ are $\phi_1(x), u_1\phi_1(x), u_2\phi_1(x)$ ie) $x, x^3/2, x^{-1}/2$

$\therefore$ The independent solution of $L(y)=0$ are x, x^3 and x^{-1}

The non-homogenous equation

Given $\phi_1(x) = x^2$ is a solution of $y'' - \frac{2}{x^2}y = x, \ (0 < x < \infty)$ Find Another independent solution

1. $y'' - \frac{2}{x^2}y = x,$

(using unit - II Problem - 1)

Solution:- $\phi_1(x) = x^2, \ \phi_2(x) = \frac{1}{x} = x^{-1}$

Hence the particular solution of $L(y)=0$ is of the form

$\psi_p = u_1\phi_1 + u_2\phi_2$

Where u_1' and u_2' satisfies the equation

$\phi_1 u_1' + \phi_2 u_2' = 0$

$\phi_1' u_1' + \phi_2' u_2' = b(x)$

$x^2 u_1' + x^{-1} u_2' = 0 \longrightarrow (1)$

$2x u_1' - x^{-2} u_2' = x \longrightarrow (2)$

$x^2 u_1' + x^{-1} u_2' = 0$

② $\times x$ $2x^2 u_1' - x^{-1} u_2' = x^2$

$3x^2 u_1' \quad = x^2$

$u_1' = x^2 / 3x^2$

$\boxed{u_1' = 1/3} \Rightarrow u_1 = \frac{1}{3} x$ 8

sub in ① $u_1' = 1/3$

$x^2/3 + x^{-1} u_2' = 0$

$x^{-1} u_2' = -x^2/3 = \frac{-x^2}{3x^{-1}} = \frac{-x^2 \cdot x}{3}$

$\boxed{u_2' = -x^3/3} \Rightarrow u_2 = \frac{-x^4}{3\times 4} = \frac{-x^4}{12}$

$y_p = \frac{x}{3} x^2 + x^{-1} \times \frac{-x^4}{12}$

$= x^3/3 + \left(\frac{-x^3}{12}\right)$

$y_p = x^3/4$ $\quad \frac{4x^3}{12} + \frac{-x^3}{12}$ $\quad y(x) = c_1 \phi_1(x) + c_2 \phi(x) + y_p$

The required equation is

$y(x) = c_1 x^2 + c_2 x^{-1} + x^3/4$

Homogenous equations with analytical coefficients

1. Find two linearly independent power series solution of $y'' - xy = 0$

solution:

Let $L(y) = y'' - xy$

then the power series solution of $L(y) = 0$ is

$\phi(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

ie) $\phi(x) = \sum_{k=0}^{\infty} c_k x^k$ $\quad \phi'(x) = c_1 + 2c_2 x + 3c_3 x^2$

$\phi'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$ $\Leftarrow k c_k x^{k-1}$

$\phi''(x) = \sum_{k=2}^{\infty} k c_k (k-1) x^{k-2}$

Since $\phi(x)$ is a solution of $L(y) = 0$

$L(\phi) = 0 \Rightarrow \phi'' - x\phi = 0$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - x \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\Rightarrow \sum_{k=2-2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0+1}^{\infty} c_k x^{k+1} = 0$$

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$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$\Rightarrow 2\cdot1\cdot c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$\Rightarrow 2\cdot1\cdot c_2 + \sum_{k=1}^{\infty} \{(k+2)(k+1) c_{k+2} - c_{k-1}\} x^k = 0$$

Hence,

$2\cdot1\cdot c_2 = 0$ and $(k+2)(k+1) c_{k+2} - c_{k-1} = 0$

$\Rightarrow c_2 = 0$ $\quad\Rightarrow (k+2)(k+1) c_{k+2} = c_{k-1}$

$$\Rightarrow c_{k+2} = \frac{1}{(k+2)(k+1)} c_{k-1} \quad \text{for } k = 1, 2, \ldots\ldots$$

For $k=1$, $c_3 = \frac{1}{3\cdot2} c_0$

For $k=2$, $c_4 = \frac{1}{4\cdot3} c_1$

For $k=3$, $c_5 = \frac{1}{5\cdot4} c_2 = 0$

For $k=4$, $c_6 = \frac{1}{6\cdot5} c_3 = \frac{1}{6\cdot5\cdot3\cdot2} c_0$

For $k=5$, $c_7 = \frac{1}{7\cdot6} c_4 = \frac{1}{7\cdot6\cdot4\cdot3} c_1$

For $k=6$, $c_8 = \frac{1}{8\cdot7} c_5 = 0$

For $k=7$, $c_9 = \frac{c_0}{9\cdot8\cdot6\cdot5\cdot3\cdot2}$

For $k=8$, $c_{10} = \frac{c_1}{10\cdot9\cdot7\cdot6\cdot4\cdot3}$

In general

For $m = 1, 2, 3$

$$c_{3m-1} = 0$$

(10)

$$c_{3m} = \frac{c_0}{3m(3m-1)(3m-3)(3m-4)\cdots 3\cdot 2}$$

and

$$c_{3m+1} = \frac{c_1}{(3m+1)3m(3m-2)(3m-3)\cdots 4\cdot 3}$$

$\therefore$ The power series solution $\phi(x)$ is

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + \cdots\cdots$$

$$= c_0 + c_1 x + \frac{c_0}{3\cdot 2}x^3 + \frac{c_1}{4\cdot 3}x^4 + \frac{c_0}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{c_1}{7\cdot 6\cdot 4\cdot 3}x^7 + \cdots\cdots$$

$$= c_0\left\{1 + \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 + \cdots\right\} + c_1\left\{x + \frac{1}{4\cdot 3}x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3}x^7 + \cdots\right\}$$

$$= c_0\left\{1 + \sum_{m=1}^{\infty}\frac{x^{3m}}{3m(3m-1)\cdots 3\cdot 2}\right\} + c_1\left\{x + \sum_{m=1}^{\infty}\frac{x^{3m+1}}{(3m+1)3m\cdots 4\cdot 3}\right\}$$

$$\phi(x) = c_0\,\phi_1(x) + c_1\,\phi_2(x)$$

where,
$$\phi_1(x) = \left\{1 + \sum_{m=1}^{\infty}\frac{x^{3m}}{3m(3m-1)\cdots 3\cdot 2}\right\}$$

$$\phi_2(x) = \left\{x + \sum_{m=1}^{\infty}\frac{x^{3m+1}}{(3m+1)3m\cdots 4\cdot 3}\right\}$$

The Legendre equation

2m The Legendre equation is

(*) $L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ where α is a constant.

Solution of a Legendre equation

Let $L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$$\therefore y'' - \frac{2x}{1-x^2} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$$

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$$\Rightarrow y'' + a_1(x)y' + a_2(x)y = 0$$

where $a_1(x) = \dfrac{-2x}{1-x^2}$ and $a_2(x) = \dfrac{\alpha(\alpha+1)}{1-x^2}$

The Functions $a_1(x)$ and $a_2(x)$ are analytic at $x=0$, since $\dfrac{1}{1-x^2} = (1-x^2)^{-1} = 1 + x^2 + x^4 + \dots$

$$= \sum_{k=0}^{\infty} x^{2k}$$

Also, this series is converges for all $|x|<1$, since $a_1(x) = -2\sum_{k=0}^{\infty} x^{2k+1}$ and

$$a_2(x) = \alpha(\alpha+1)\sum_{k=0}^{\infty} x^{2k}$$

which is converges for $|x|<1$

To find the basis for the solution

Let ϕ be the solution of the legendre equation on $|x|<1$

$$\text{Let } \phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\Rightarrow \phi'(x) = \sum_{k=1}^{\infty} k C_k x^{k-1}$$

$$\Rightarrow \phi''(x) = \sum_{k=2}^{\infty} k(k-1) C_k x^{k-2}$$

Since $\phi(x)$ is a solution of $L(y)=0$

$$L(\phi) = 0$$

$$\Rightarrow (1-x^2)\phi'' - 2x\phi' + \alpha(\alpha+1)\phi = 0$$

$$\Rightarrow (1-x^2)\sum_{k=2}^{\infty} k(k-1) C_k x^{k-2} - 2x\sum_{k=1}^{\infty} k C_k x^{k-1} + \alpha(\alpha+1)\sum_{k=0}^{\infty} C_k x^k = 0$$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1) c_k x^k - \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} \alpha(\alpha+1) c_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) c_k x^k$$

$$- \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} \alpha(\alpha+1) c_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=0}^{\infty} k(k-1) c_k x^k - \sum_{k=0}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} \alpha(\alpha+1) c_k x^k = 0$$

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$$\Rightarrow \sum_{k=0}^{\infty} \left\{ (k+2)(k+1) c_{k+2} - c_k \{ k(k-1) + 2k - \alpha(\alpha+1) \} \right\} x^k = 0$$

$$(k+2)(k+1) c_{k+2} - c_k \{ k^2 - k + 2k - \alpha(\alpha+1) \} = 0$$

$$\Rightarrow (k+2)(k+1) c_{k+2} = c_k \{ k^2 + k - \alpha(\alpha+1) \}$$

$$c_{k+2} = \frac{\{ k^2 + k - \alpha(\alpha+1) \}}{(k+2)(k+1)} c_k, \quad \text{for } k = 0, 1, 2, \ldots$$

$$= \frac{k^2 + k + \alpha k - \alpha k - \alpha^2 - \alpha}{(k+1)(k+2)} c_k$$

$$= \frac{k(\alpha + k + 1) - \alpha(\alpha + k + 1)}{(k+1)(k+2)} c_k$$

$$c_{k+2} = \frac{-(\alpha + k + 1)(\alpha - k)}{(k+1)(k+2)} c_k \quad \text{for } k = 0, 1, 2, \ldots$$

put $k=0$, $c_2 = \dfrac{-(\alpha+1)(\alpha)}{1 \cdot 2} c_0$

put $k=1$, $c_3 = \dfrac{-(\alpha+2)(\alpha-1)}{2 \cdot 3} c_1$

put $k=2$, $c_4 = \dfrac{-(\alpha+3)(\alpha-2)}{3 \cdot 4} c_2 = \dfrac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{1 \cdot 2 \cdot 3 \cdot 4} c$

Put $k=3$, $C_5 = \frac{-(\alpha+4)(\alpha-3)}{4\cdot 5} C_3$

$= \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5\cdot 4\cdot 3\cdot 2} C_1$

In general,

$$C_{2m} = \frac{(-1)^m (\alpha+2m-1)\cdots(\alpha+1)(\alpha)(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!} C_0$$

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$$C_{2m+1} = \frac{(-1)^m (\alpha+2m)\cdots(\alpha+2)(\alpha-1)\cdots(\alpha-2m-1)}{(2m+1)!} C_1$$

Hence,

$$\phi(x) = C_0 + C_1 x - \frac{(\alpha+1)(\alpha)}{2!} C_0 x^2 - \frac{(\alpha+2)(\alpha-1)}{3!} C_1 x^3$$

$$+ \frac{(\alpha+3)(\alpha+1)(\alpha)(\alpha-2)}{4!} C_0 x^4 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha}{5!}$$

$$\Rightarrow \phi(x) = C_0 \left\{ 1 - \frac{(\alpha+1)\alpha}{2!} x^2 + \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4!} x^4 + \cdots \right.$$

$$+ C_1 \left\{ x - \frac{(\alpha+2)(\alpha-1)}{3!} x^3 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5!} \right.$$

$$\Rightarrow \phi(x) = C_0 \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m-1)\cdots(\alpha+1)\alpha(\alpha-2\cdots(\alpha-2m)}{(2m)!} \right.$$

$$+ C_1 \left\{ x + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m)\cdots(\alpha+2)(\alpha-1)\cdots}{(2m+1)!} \right.$$

$\Rightarrow \phi(x) = C_0\, \phi_1(x) + C_1\, \phi_2(x)$ where

$$\phi_1(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m-1)\cdots(\alpha+1)(\alpha+2)\cdots(\alpha-2}{(2m)!}$$

and $$\phi_2(x) = x + \sum_{m=1}^{\infty} \frac{(-1)^m (\alpha+2m)\cdots(\alpha+2)(\alpha-1)\cdots(}{(2m+1)!}$$

Both $\phi_1(x)$ and $\phi_2(x)$ are solutions of the legendre equation those corresponding to the choices $c_0=1, c_1=0$ and $c_0=0, c_1=1$ respectively.

Also, since $\phi_1(0)=1$, $\phi_2(0)=0$ and $\phi_1'(0)=0$, $\phi_2'(0)=1$. They form a basis for the solution

Note: If $\alpha=0$; $\phi_1(x)=1$ (14)

If $\alpha=2$; $\phi_1(x)=1-\frac{3\cdot 2}{2}x^2$

If $\alpha=4$, $\phi_1(x)=1-\frac{5\cdot 4}{2}x^2+\frac{7\cdot 5\cdot 4\cdot 2}{4!}x^4$

$$=1-10x^2+\frac{35}{3}x^4$$

If $\alpha=1$; $\phi_2(x)=x$

If $\alpha=3$; $\phi_2(x)=x-\frac{5\cdot 2}{3!}x^3$

If $\alpha=5$: $\phi_2(x)=x-\frac{7\cdot 4}{3!}x^3+\frac{9\cdot 7\cdot 4}{5!}x^5$

$$=x-\frac{14}{3}x^3+\frac{21}{5}x^5$$

Definition:

The polynomial solutions of degree n of $(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0$ satisfying $\boxed{P_n(1)=1}$ is called the n^{th} legendre polynomial $P_n(1)=1$

To find the polynomial solution $P_n(x)$

Let ϕ be the polynomial of degree n defined by
$$\boxed{\phi(x)=\frac{d^n}{dx^n}(x^2-1)^n}$$

$\therefore$ ϕ satisfies the legendre equation
$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0$$

Let $u(x) = (x^2-1)^n$

Differentiate with respect to x,

$$u'(x) = n(x^2-1)^{n-1} \cdot 2x$$

$$u'(x) = 2nx\,(x^2-1)^{n-1}$$

$$\Rightarrow (x^2-1)\,u'(x) = 2nx\,(x^2-1)^n \Rightarrow 2nx\,u(x)$$

$$\Rightarrow (x^2-1)u'(x) - 2nx\,u(x) = 0 \qquad (15)$$

Differentiate with respect to 'x'

$$(x^2-1)u''(x) + u'(x)2x - 2nx\,u'(x) - 2nu(x) = 0$$

Diff $(n+1)^{th}$ time we get

$$(x^2-1)u^{(n+2)}(x) + 2x(n+1)u^{(n+1)} + (n+1)nu^{(n)} - 2nx\,u^{(n+1)} - 2n(n+1)u^{(n)} = 0$$

since $\phi = [u(x)]^{(n)}$ we get

$$(x^2-1)\phi'' + 2x(n+1)\phi' + (n+1)n\phi - 2nx\phi' - 2n(n+1)\phi = 0$$

$$\Rightarrow (x^2-1)\phi'' + 2x(n+1-n)\phi - n(n+1)\phi = 0$$

$$\Rightarrow (x^2-1)\phi'' + 2x\phi' - n(n+1)\phi = 0$$

$$\Rightarrow (1-x^2)\phi'' - 2x\phi' + n(n+1)\phi = 0$$

$\therefore \phi$ satisfies the legendre equation

Now, $\phi(x) = [(x^2-1)^n]^{(n)}$

$$= [(x+1)^n(x-1)^n]^{(n)}$$

$$= (x+1)^n[(x-1)^n]^{(n)} + \text{terms with } (x-1) \text{ as a factor}$$

$\therefore \phi(x) = (x+1)^n\, n! +$ terms with $(x-1)$ as a factor

At $x=1$, $\phi(1) = 2^n n!$

The function $P_n(x)$ is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$ is the legendre Polynomial

Remark:-

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

2m

(K) put $n=1$, we get

$$P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2} 2x$$

$$\boxed{P_1(x) = x}$$

put $n=2$; we get

$$P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2$$

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$$= \frac{1}{8}\left\{\frac{d}{dx} 2(x^2-1)2x\right\}$$

$$= \frac{1}{2}\frac{d}{dx}(x^3-x)$$

$$\boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$$

put $n=3$; we get

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{8\times 6}\frac{d^2}{dx^2}(3(x^2-1)^2 2x)$$

$$= \frac{1}{8\times 6}\frac{d^2}{dx^2}(6x(x^2-1)^2)$$

$$= \frac{1}{8}\frac{d^2}{dx^2}(x(x^2-1)^2)$$

$$= \frac{1}{8}\frac{d}{dx}((x^2-1)^2+2x(x^2-1)2x)$$ $x(x^2-1)^2$

$(x^2-1)^2(1)+x\,2(x^2-1)(2x)$

$$= \frac{1}{8}\frac{d}{dx}(x^2-1)^2+4x^4-4x^2$$

$$= \frac{1}{8}\{2(x^2-1)(2x)+16x^3-8x\}$$

$$= \frac{1}{2}(x^3-x+4x^3-2x)$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

2m (H)

$$= \frac{1}{16\times 24}\frac{d^4}{dx^4}\{x^8-4x^6+6x^4-4x^2+1\}$$

$$= \frac{1}{16 \times 24} \frac{d^3}{dx^3}\{8x^7 - 24x^5 + 24x^3 - 8x\}$$

$$= \frac{8}{16 \times 24} \frac{d^3}{dx^3}\{x^7 - 3x^5 + 3x^3 - x\}$$

$$= \frac{1}{48} \frac{d^2}{dx^2}\{7x^6 - 15x^4 + 9x^2 - 1\}$$

$$= \frac{1}{48} \frac{d}{dx}\{42x^5 - 60x^3 + 18x\}$$

$$= \frac{6}{48} \frac{d}{dx}\{7x^5 - 10x^3 + 3x\} \quad 17$$

$$P_4(x) = \frac{1}{8} \frac{d}{dx}\{7x^5 - 10x^3 + 3x\} = \frac{1}{8}\{35x^4 - 30x^2 + 3\}$$

$$\therefore P_4(x) = \frac{1}{8}\{35x^4 - 30x^2 + 3\}$$

---

(1) Show that $\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$, $(n \neq m)$ APR 18

Proof: we know that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

since the legendre polynomial, $P_n(x)$ satisfies the legendre equation, $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ $[\because y = P_n]$

$$\Rightarrow (1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n(x) = 0$$

$$\Rightarrow (1-x^2)P_n''(x) - 2x P_n'(x) = -n(n+1)P_n(x)$$

uv format

$$\Rightarrow [(1-x^2)P_n'(x)]' = -n(n+1)P_n(x) \rightarrow (1)$$

Similarly

$$[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x) \rightarrow (2)$$

$$(1) \times P_m(x) \Rightarrow P_m(x)[(1-x^2)P_n'(x)]' = -n(n+1)P_m(x)P_n(x) \rightarrow (3)$$

$$(2) \times P_n(x) \Rightarrow P_n(x)[(1-x^2)P_m'(x)]' = -m(m+1)P_m(x)P_n(x) \rightarrow (4)$$

$$(3) - (4) \Rightarrow P_m(x)[(1-x^2)P_n'(x)]' - P_n(x)[(1-x^2)P_m'(x)]' = -n(n+1)P_m(x)P_n(x) + m(m+1)P_m(x)P_n(x)$$

common

$\Rightarrow P_m(x)P_n(x)\{m(m+1)-n(n+1)\} =$

$P_m(x)(1-x^2)P_n''(x) + P_m(x)P_n'(x)(1-x^2)' - P_n(x)(1-x^2)P_m''(x) - P_n(x)P_m'(x)(1-x^2)'$

$= (1-x^2)[P_n''(x)P_m(x) - P_n(x)P_m''(x)] + (1-x^2)'[P_m(x)P_n'(x) - P_n(x)P_m'(x)]$

$= (1-x^2)[P_n''(x)P_m(x) + P_n'(x)P_m'(x) - P_n'(x)P_m'(x) - P_n(x)P_m''(x)] + (1-x^2)'[P_m(x)P_n'(x) - P_n(x)P_m'(x)]$

$= (1-x^2)[P_m(x)P_n'(x) - P_n(x)P_m'(x)]' + (1-x^2)'[P_m(x)P_n'(x) - P_n(x)P_m'(x)]$

$= [(1-x^2)[P_m(x)P_n'(x) - P_n(x)P_m'(x)]]'$

Integrating

$\{m(m+1)-n(n+1)\}\int_{-1}^{1} P_n(x)P_m(x)dx$

$= [(1-x^2)\{P_m(x)P_n'(x) - P_n(x)P_m'(x)\}]_{-1}^{1}$

$= 0$

$\therefore \int_{-1}^{1} P_n(x)P_m(x)dx = 0$, provided $n \neq m$

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(2) Show that $\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$ Apr-19

Proof:-

(3) We know that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

(4) Let $u(x) = (x^2-1)^n$

then $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}u(x)$

Claim : $u^{(k)}(1) = u^{(k)}(-1) = 0$; if $0 \le k < n$

Now,

$u(x) = (x^2-1)^n$

$\Rightarrow u'(x) = n(x^2-1)^{n-1}\cdot 2x$

$\Rightarrow u'(x) = 2nx \; (x^2-1)^{n-1}$ $\quad 2nx(n-1)(x^2-1)^{n-2}\,2x = 4nx^2(n-1)(x^2-1)$

$\Rightarrow u''(x) = 4nx^2(n-1)(x^2-1)^{n-2} + 2n(x^2-1)^{n-1}$

$\Rightarrow u''(x) = (x^2-1)^{n-2}\{4nx^2(n-1) + 2n(x^2-1)\}$

$\Rightarrow u''(x) = (x^2-1)^{n-2}\{4n(n-1)x^2 + 2nx^2 - 2n\}$

In general,

$$u^{(n-1)}(x) = (x^2-1)^{n-(n+1)}\{\text{polynomial of degree } n-1\}$$

$$u^{(n-1)}(x) = (x^2-1)\{\text{polynomial of degree } n-1\}$$

Since $u^{(k)}(x)$; $k = 0, 1, 2, \ldots (n-1)$ is a multiple of (x^2-1);

$$u^{k}(1) = u^{(k)}(-1) = 0 \quad ; \text{ if } 0 \le k \le n$$

Now, $\int_{-1}^{1} u^{(n)}x \; v^{(n)}x \, dx = \left[u^{(n-1)}(x)\, u^{(n)}(x) \right]_{-1}^{1} - \int_{-1}^{1} u^{(n-1)}(x)\, u^{(n+1)}(x)\,dx$

$\int uv\,dv = uv - \int v\,du$

$$= -\int_{-1}^{1} u^{(n-1)}(x)\; u^{(n+1)}(x)\,dx$$

$dv = v^{(n)} \quad = v^{(n-1)}$

$$= (-1)^2 \int_{-1}^{1} u^{(n-2)}(x)\; u^{(n+2)}(x)\,dx$$

$\vdots$

$$= (-1)^n \int_{-1}^{1} u^{(0)}(x)\; u^{(2n)}(x)\,dx$$

$[(-1)(x^2-1)]^n = (1-x^2)^n$

$$= (-1)^n \int_{-1}^{1} (x^2-1)^n (2n)!\,dx$$

$$= (-1)^n (2n)! \int_{-1}^{1} (x^2-1)^n dx = (2n)! \int_{-1}^{1} (1-x^2)^n dx \Rightarrow$$

let $x = \sin\theta$

$\Rightarrow dx = \cos\theta\, d\theta$

If $x = 1 \Rightarrow \theta = \pi/2$

If $x = -1 \Rightarrow \theta = -\pi/2$

$$\int_{-1}^{1} (1-x^2)^n dx = \int_{-\pi/2}^{\pi/2} (\cos^2\theta)^n \cos\theta\, d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \cos^{2n+1}\theta \, d\theta = 2\int_0^{\pi/2} \cos^{2n+1}\theta \, d\theta$$

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 2}{n(n-2)(n-4)\cdots 3} \quad n \text{ is odd}$$

$$= 2\left(\frac{2n+1-1}{2n+1}\right)\left(\frac{2n+1-3}{2n+1-2}\right)\cdots \frac{2}{3}$$

$$= 2\,\frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3} \times \frac{(2n)(2n-3)\cdots 4\cdot 2}{(2n)(2n-3)\cdots 4\cdot 2}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)(n-5)\cdots 1}{n(n-2)(n-4)\cdots 2} \times \frac{\pi}{2} \quad n \text{ is even}$$

$$= \frac{2}{(2n+1)!}\left[2n(2n-2)\cdots 4\cdot 2\right]^2$$

$$= \frac{2}{(2n+1)!}\left[2^n n(n-1)\cdots 2\cdot 1\right]^2$$

$$= \frac{2}{(2n+1)!}\, 2^{2n}(n!)^2$$

(20)

$\therefore$ Equation Ⓐ becomes

$$\int_{-1}^{1} u^{(n)}(x)\, u^{(n)}x = (2n)!\,\frac{2}{(2n+1)!}\,2^{2n}(n!)^2$$

$[2^n(n!)]^2$

$\div (2^n n!)^2$

$$\therefore \frac{1}{[2^n(n!)]^2}\int_{-1}^{1} u^{(n)}(x)\, u^{(n)}(x)\,dx$$

$$= \frac{1}{(2^n n!)^2}\,(2n)!\,\frac{2}{(2n+1)!}\,2^{2n}(n!)^2$$

$$= \frac{2}{2n+1}$$

$$\therefore \int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$$

Ⓐ ③ Prove that $P_n(-x) = (-1)^n P_n(x)$ 2m AP-18

Proof: We know that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

We prove the result by induction on n

If $n=1$ $\quad P_1(x) = \frac{1}{2\cdot 1!}\frac{d}{dx}(x^2-1)$

Given condition

Let $t = -x \Rightarrow dt = -dx$

$$\therefore \frac{d}{dt}(t^2-1) = 2t = -2x$$
$$= -\frac{d}{dx}(x^2-1)$$

$$\tfrac{1}{2}\frac{d}{dt}(t^2-1) = -\tfrac{1}{2}\frac{d}{dx}(x^2-1)$$

$$P_1(t) = -P_1(x)$$ 21

$$\Rightarrow P_1(-x) = -P_1(x)$$

We assume that the result is true for $n=k$

ie) $P_k(-x) = (-1)^k P_k(x)$.

we prove this result for $n=k+1$

It is enough to prove that.

$$\frac{d^{k+1}}{dt^{k+1}}(t^2-1)^{k+1} = (-1)^{k+1}\frac{d^{k+1}}{dx^{k+1}}(x^2-1)$$

Now, $$\frac{d^{k+1}}{dt^{k+1}}(t^2-1)^{k+1} = \frac{d^k}{dt^k}\left[\frac{d}{dt}(t^2-1)^{k+1}\right]$$

$$= \frac{d^k}{dt^k}\left\{(k+1)(t^2-1)^k 2t\right\}$$

$$= (-1)^k\frac{d^k}{dx^k}\left\{(k+1)2(-x)(x^2-1)^k\right\}$$

$$= (-1)^{k+1}\frac{d^k}{dx^k}\left\{[(x^2-1)^{k+1}]'\right\}$$

$$= (-1)^{k+1}\frac{d^{k+1}}{dx^{k+1}}\left[(x^2-1)^{k+1}\right]$$

Hence $P_n(-x) = (-1)^n P_n(x)$

---

(7) Show that $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

Proof:

$$(1-2xz+z^2)^{-1/2} = (1-z(2x-z))^{-1/2}$$

$(1-x)^n = 1+nx+\frac{n(n+1)}{1\cdot 2}x^2\ldots$

$$= 1+\frac{1}{2}(z(2x-z)) + \frac{\frac{1}{2}(\frac{1}{2}+1)}{1\cdot 2}(z(2x-z))^2$$
$$+ \frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{1\cdot 2\cdot 3}(z(2x-z))^3 + \cdots$$

$$\left(\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \cdots\right)$$

$$= 1 + \frac{1}{2} z(2x-z) + \frac{1}{2}\cdot\frac{1}{2}\left(\frac{3}{2}\right) z^2(2x-z)^2 + \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(z(2x-z))^3 + \cdots\cdots$$

$$= 1 + xz - \frac{1}{2}z^2 + \frac{3}{8}z^2(4x^2+z^2-4xz) + \frac{15}{48}(z^3(8x^3-4x^2z+2xz^2-z^3) + \cdots\cdots$$

$$= 1 + xz - \frac{1}{2}z^2 + 3/_2\, x^2z^2 + \frac{3}{8}z^4 - 3/_2\, xz^3 + \frac{15}{6}x^3z^3 - \frac{15}{12}x^2z^4 + \frac{15}{24}xz^5 - \frac{15}{48}z^3 + \cdots$$

$$= 1 + xz + \frac{1}{2}(3x^2-1)z^2 + 1/_2\,(5x^3-3x)z^3 + \cdots\cdots$$

$$= 1 + P_1(x)z + P_2(x)z^2 + P_3(x)z^3 + \cdots$$

Remark

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$$\left(\because P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1)\right.$$ and $$\left.P_3(x) = \frac{1}{2}(5x^3-3x)\right)$$

$$\therefore (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n$$

(5) Prove that the coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n(n!)^2}$

NOV-18

Solution:-

We know that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$$\therefore P_n(x) = \frac{1}{2^n n!}\left[(x^2-1)^n\right]^{(n)} = \frac{1}{2^n n!}\left((x-1)^n(x+1)^n\right)^{(n)}$$

$$= \frac{1}{2^n n!}\Big\{(x+1)^n\left[(x-1)^n\right]^{(n)} + nC_1\left[(x+1)^n\right]^{(1)}\left[(x-1)^n\right]^{(n-1)}$$

$$+ nC_2\left[(x+1)^n\right]^{(2)}\left[(x-1)^n\right]^{(n-2)} + \cdots + nC_n\left[(x+1)^n\right]^{(n)}(x-1)^n\Big\}$$

$x^n + nC_1 x^{n-1}$

$\therefore$ The coefficient of x^n in $P_n(x)$ is

Unit - III

Linear equation with regular singular point

Definition :-

A point x_0 such that $a_0(x_0)=0$ is called a singular point of the equation

$x = x_0$

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0$$

We say that x_0 is a regular singular point for $a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_n(x)y = 0$ of the equation can be written in the form.

$$(x-x_0)^n y^{(n)} + b_1(x)(x-x_0)^{n-1}y^{(n-1)} + \dots + b_n(x)y = 0$$

near x_0 where the functions $b_1, b_2, \dots b_n$ are analytic at x_0

$x^2=0 \Rightarrow x=0$ singular

Ex:- Consider the equation $x^2y'' - y' - \frac{3}{4}y = 0$

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$\alpha\ (n!)^2$

Unit – III

1

Linear equation with regular singular point

Definition :-

A point x_0 such that $a_0(x_0)=0$ is called a singular point of the equation

$x = x_0$

$$a_0(x)y^{(n)}+a_1(x)y^{(n-1)}+\dots+a_n(x)y=0$$

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$$(x-x_0)^n y^{(n)}+b_1(x)(x-x_0)^{n-1}y^{(n-1)}+\dots+b_n(x)y=0$$

near x_0 where the functions $b_1, b_2, \dots b_n$ are analytic at x_0

$x^2=0 \to x=0$ singular pt

Ex:- Consider the equation $x^2y''-y'-\frac{3}{4}y=0$

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3 The origin $x_0 = 0$ is a singular point but not a regular singular point, since $a_1(x) = -1$ which is not of the form $x\, b_1(x)$ where b_1 is analytic at 0

$(x-0)^2 y'' - 1$ ... not a regular singular pt

Euler equation

Definition

A second order equation having a regular singular point at origin is the Euler equation

$$L(y) = x^2 y'' + a x y' + b y = 0$$

where a, b are constants

Theorem: 3.1

Consider the second order Euler equation $x^2 y'' + a x y' + b y = 0$ (a, b constants) and the polynomial q given by $q(r) = r(r-1) + ar + b$

A basis for the solution of the Euler equation on any interval not containing $x = 0$ is given by $\phi_1(x) = |x|^{r_1}$ and $\phi_2(x) = |x|^{r_2}$ incase r_1, r_2 are distinct roots of q and

$$\phi_1(x) = |x|^{r_1} ;\ \phi_2(x) = |x|^{r_1} \log|x|, \text{ if } r \text{ is a root}$$

of q of multiplicity two

Example:- Type-I

Consider the equation $x^2 y'' + x y' + y = 0$, $x \neq 0$

$$x^2 y'' + x y' + y = 0$$

$$q(r) = r(r-1) + r + 1$$

UQ NOV-17 (2)

$$= r^2 - r + r + 1$$

$$q(r) = 0 \Rightarrow r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm i$$

$$r_1 = i,\ r_2 = -i$$

$$\phi_1(x) = |x|^{i} ;\ \phi_2(x) = |x|^{-i}$$

$$\therefore \phi(x) = c_1 |x|^{i} + c_2 |x|^{-i}$$

where c_1 c_2 are constants

which is the required solution

Definition:-

The euler equation of the n^{th} order is of the form $L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \dots + a_n y = 0$

where $a_1, a_2, \dots a_n$ are constants

Definition :- The polynomial $q(r)$ of degree n

$q(r) = r(r-1)\cdots(r-(n+1)) + a_1 r(r-1)\cdots(r-n+2)+\ldots + a_n$

is called the indicial polynomial

Further euler equation

$L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0$

APr-19

③

① Find all solutions of the following equation for $x > 0$

$x^2 y'' + 2xy' - 6y = 0$

UQV NOV-18 Let $L(y) = x^2 y'' + 2xy' - 6y = 0$

NOV-15 $q(r) = r(r-1) + 2r - 6$

$r^2 - r + 2r - 6$

$q(r) = 0 \Rightarrow r^2 + r - 6 = 0$

$\Rightarrow (r+3)(r-2) = 0$

$\Rightarrow r = -3 \; ; r = 2$

The solution of $L(y) = 0$ are

$\phi_1(x) = x^{-3} \; ; \phi_2(x) = x^2$

$\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$

$\phi(x) = c_1 x^{-3} + c_2 x^2$

where c_1, c_2 are constants

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② Find all solutions of the following equation for $x > 0$

$2x^2 y'' + xy' - y = 0$

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Let $L(y) = x^2 y'' + \frac{1}{2} xy' - \frac{1}{2} y = 0$

$q(r) = r(r-1) + \frac{1}{2} r - \frac{1}{2}$

$= r^2 - r + \frac{r}{2} - \frac{1}{2}$

$= r^2 - r/2 - \frac{1}{2}$

$q(r) = 0 \Rightarrow r^2 - r/2 - \frac{1}{2} = 0$

$\Rightarrow 2r^2 - r - 1 = 0$

$\Rightarrow 2r^2 - 2r + r - 1 = 0$

$\Rightarrow 2r(r-1) + r - 1 = 0$

$\Rightarrow (r-1)(2r+1) = 0$

$\Rightarrow r = 1, \; r = -\frac{1}{2}$

$r = 1, -1/2$

The solution of $L(y)=0$ are

$\phi_1(x) = x^1$ and $\phi_2(x) = x^{-1/2}$

$\therefore \phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$

$\phi(x) = c_1 x + c_2 x^{-1/2}$

where c_1, c_2 are constants

③ $x^2y'' - 3xy' + 5y = 0 \quad |x|>0$

solution Let $L(y) = x^2y'' - 3xy' + 5y = 0$

$q(r) = r(r-1) - 3r + 5$

$= r^2 - r - 3r + 5$

$= r^2 - 4r + 5$

$q(r) = 0 \Rightarrow r^2 - 4r + 5 = 0$

$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $r = \frac{4 \pm \sqrt{16 - 4(5)}}{2} = \frac{4 \pm \sqrt{16-20}}{2}$

$= \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2}$

$r = 2+i, 2-i$

The solution of $L(y)=0$ are

$\phi_1(x) = |x|^{2+i}$; $\phi_2(x) = |x|^{2-i}$

$\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$

$= c_1 |x|^{2+i} + c_2 |x|^{2-i}$

where c_1, c_2 are constants,

4 Find all solutions of the following equations for $x>0$

1) $x^3y''' + 2x^2y'' - xy' + y = 0$

solution: Let $L(y) = x^3y''' + 2x^2y'' - xy' + y = 0$

$q(r) = r(r-1)(r-2) + 2r(r-1) - r + 1$

$= (r^2 - r)(r-2) + 2(r^2 - r) - r + 1$

$= r^3 - 2r^2 - r^2 + 2r + 2r^2 - 2r - r + 1$

$= r^3 - r^2 - r + 1$

$q(r) = 0 \Rightarrow r^3 - r^2 - r + 1 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -1 & 1 \\ & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

$(r-1)(r^2-1)=0$

$r^2=1$

$r=\pm 1$

$r=1,-1,1$

$r_1=1 \;;\; r_2=1 \;;\; r_3=-1$

The solutions of $L(y)=0$ are

$\phi_1(x) = |x|^1$

$\phi_2(x) = |x| \log|x|$

$\phi_3(x) = |x|^{-1}$

$\phi(x) = c_1 x + c_2 x \log x + c_3 x^{-1}$

where c_1 c_2 and c_3 are constants.

5. $x^2y'' - (2+i)xy' + 3iy = 0$, $|x|>0$

solution $L(y) = x^2y'' - (2+i)y' + 3iy = 0$

$q(r) = r(r-1) - (2+i)r + 3i$

$= r^2 - r - 2r - ir + 3i$

$= r^2 - 3r - ir + 3i$

$q(r)=0 \Rightarrow r^2 - 3r - ir + 3i = 0$

$\Rightarrow r^2 - (3+i)r + 3i = 0$

$\Rightarrow r = 3, i$

The solutions $L(y)=0$ are

$\phi_1(x) = 3 \;;\; \phi_2(x) = i$

$\phi(x) = c_1\phi_1(x) + c_2\phi_2(x)$

$= c_1|x|^3 + c_2|x|^i$

where c_1, c_2 are constants

6. $x^2y'' + xy' - 4y = x$; $x>0$

solution: Let $L(y) = x^2y'' + xy' - 4y = 0$

The indicial polynomial is

$q(r) = r(r-1) + r - 4$

$= r^2 - r + r - 4$

$= r^2 - 4$

$q(r)=0 \Rightarrow r^2 - 4 = 0$

$r^2 = 4$

$r = \pm 2$

The solution of $L(y)=0$ are

$\phi_1(x) = x^2 \; ; \; \phi_2(x) = x^{-2}$

$C.F = c_1 x^2 + c_2 x^{-2}$

The particular Integral of the given equation is of the form

$y = Ax + B$

6

sub given eqn

$y' = A$

$y'' = 0$

$\therefore x^2(0) + x(A) - 4(Ax + B) = x$

$\Rightarrow Ax - 4Ax - 4B = x$

$\Rightarrow -3Ax - 4B = x$

Equating the coefficients x and constants

$-3A = 1$

$\boxed{A = -1/3}$

$-4B = 0$

$\boxed{B = 0}$

$P.I = y = Ax + B = -x/3$

$\phi(x) = C.F + P.I$

$= c_1 x^2 + c_2 x^{-1} - x/3$

7 $x^2 y'' + xy' + 4y = 1 \; ; \; |x| > 0$

solution Let $L(y) = x^2 y'' + xy' + 4y = 0$

$q(r) = r(r-1) + r + 4$

$= r^2 - r + r + 4$

$= r^2 + 4$

$q(r) = 0 \Rightarrow r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

The solution $L(y) = 0$ are

$\phi_1(x) = |x|^{2i}$

$\phi_2(x) = |x|^{-2i}$

$C.F = c_1 |x|^{2i} + c_2 |x|^{-2i}$

The particular integral of the given equation of the form

$y = Ax + B$

$y' = A$

$y'' = 0$

$\therefore x^2 y'' + xy' + 4y = 1$

$\Rightarrow x^2(0) + x(A) + 4Ax + 4B = 1$

$5Ax + 4B = 1$

Equating coefficients x and constants

$5Ax = 0$

$A = 0$

7

$4B = 1$

$B = 1/4$

P.I $= Ax + B$
$0 + 1/4$

$P.I = y = 1/4$

$\phi(x) = C.F + P.I$

Hence the required solution is

$\phi(x) = c_1 |x|^{2i} + c_2 |x|^{-2i} + 1/4$

---

8 $x^2y'' + xy' - 4\pi y = x$; $|x| > 0$

solution let $L(y) = x^2y'' + xy' - 4\pi y = 0$

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$q(r) = r(r-1) + r - 4\pi$

$= r^2 - r + r - 4\pi$

$= r^2 - 4\pi$

$q(r) = 0 \Rightarrow r^2 - 4\pi = 0$

$r^2 = 4\pi$

$r = \pm 2\sqrt{\pi}$

$\phi_1(x) = |x|^{2\sqrt{\pi}}$ $\quad \phi_2(x) = |x|^{-2\sqrt{\pi}}$

The solution of $L(y) = 0$ are

$\phi_1(x) = |x|^{2\sqrt{\pi}}$ and $\phi_2(x) = |x|^{-2\sqrt{\pi}}$

$C.F = c_1 |x|^{2\sqrt{\pi}} + c_2 |x|^{-2\sqrt{\pi}}$

The particular integral of the given equation is of the form $y = Ax + B$.

$y' = A$

$y'' = 0$

$x^2y'' + xy' - 4\pi y = x$

$x^2(0) + x(A) - 4\pi(Ax + B) = x$

$Ax - 4\pi Ax - 4\pi B = x$

$A - 4\pi A = 1$

$A(1 - 4\pi) = 1$

$$A = \frac{1}{1-4\pi}$$

where $-4\pi B = 0$

$\therefore B = 0$

$$y = \frac{x}{1-4\pi}$$

(8)

$$\phi(x) = C.F + P.I$$

$$= c_1 |x|^{2\sqrt{\pi}} + c_2 |x|^{-2\sqrt{\pi}} + \frac{x}{1-4\pi}$$

(T) U.V 10mark type-II

Compute the indicial polynomial and their roots of the following equation $x^2y'' + (x+x^2)y' - y = 0$

Solution: Let $L(y) = x^2y'' + (x+x^2)y' - y = 0$

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if $x^2 = 0 \Rightarrow x = 0$

$x = 0$ is a singular point of $L(y) = 0$

comparing the given equation with

$$(x-x_0)^2 y'' + (x-x_0)\, a(x) y' + b(x) y = 0$$

At $x_0 = 0$ $\quad x^2y'' + x\, a(x) y' + b(x) y = 0 \rightarrow (1)$

$$L(y) = 0 \Rightarrow x^2y'' + (x+x^2)y' - y = 0$$

$$\Rightarrow x^2y'' + x(1+x)y' - y = 0 \rightarrow (2)$$

$a(x) = 1 + x$, $b(x) = -1$ analytic at 0

$x = 0$ is a regular singular point of $L(y) = 0$

At $x = 0$

$a(x) = 1 + x \Rightarrow 1$

$b(x) = -1 \Rightarrow -1$

$\therefore$ The indicial polynomial

$$q(r) = r(r-1) + ar + b$$

$$= r^2 - r + r - 1$$

$$= r^2 - 1$$

$$q(r) = 0 \Rightarrow r^2 - 1 = 0$$

$$r^2 = 1$$

$$r = \pm 1$$

$\therefore$ The roots of indicial polynomial are 1 and -1

10 compute the indicial polynomial and their root for the following equation

$$x^2y'' + xy' + (x^2 - 1/4)y = 0$$ 2m APr-18

solution.

Let $L(y) = x^2y'' + xy' + (x^2 - 1/4)y$ → (1)

if $x^2 = 0 \Rightarrow x = 0$

$x = 0$ is a singular point of $L(y) = 0$

9

Comparing the given equation

$$(x - x_0)^2 y'' + (x - x_0)\, a(x) y' + b(x) y = 0$$

$$x^2 y'' + x\, a(x) y' + b(x) y = 0 \rightarrow (2)$$

$a(x) = 1$, $b(x) = x^2 - \frac{1}{4}$ analytic at 0

$x = 0$ is regular singular point of $L(y) = 0$

At $x = 0$,

$a(x) = 1 \Rightarrow a = 1$

$b(x) = x^2 - 1/4 \Rightarrow b = -1/4$

The indical polynomial

$$q(r) = r(r-1) + ar + b$$

$$= r(r-1) + r - 1/4$$

$$= r^2 - r + r - 1/4$$

$$= r^2 - 1/4$$

$q(r) = 0 \Rightarrow r^2 - 1/4 = 0$, $r^2 = 1/4$

$$r = \pm 1/2$$

The roots of indical polynomial

$1/2$, and $-1/2$

11) compute the indical polynomial and their roots for the following equation

$$x^2 y'' + (\sin x) y' + (\cos x) y = 0$$

solution:

Let $L(y) = x^2 y'' + (\sin x) y' + (\cos x) y = 0$

$x^2 = 0 \Rightarrow x = 0$

$x = 0$ is a singular point $L(y) = 0$

comparing with the given equation

$$(x-x_0)^2 y'' + (x-x_0)\, a(x) y' + b(x) y = 0$$

$$x^2 y'' + x a(x) y' + b(x) y = 0$$

$L(y)=0 \Rightarrow x^2 y'' + (\sin x) y' + (\cos x) y = 0$

since, $\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$

10

$= x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)$

$\cos x = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$

$\therefore L(y)=0$

$\Rightarrow x^2 y'' + x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right) y' + (\cos x) y = 0$

$\Rightarrow x^2 y'' + x\,(a x) y' + (b(x))\, y = 0$

where $a(x) = \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots\right)$

$b(x) = \cos x$ are analytic at 0

hence $x=0$ is a regular singular point of

$L(y)=0$ at $x=0$

$a(x)=1$, $b(x)=1$

$a=1$, $b=1$

The indicial polynomial root

$q(r) = r(r-1) + ar + b$

$= r^2 - r + r + 1$

$q(r) = 0 \Rightarrow r^2 + 1 = 0$

$r = \pm i$

The indicial polynomial roots is i and $-i$

12 Find the singular point of the following equation and determine these which are regular point $x^2 y'' + (x+x^2) y' - y = 0$

solution

Let $L(y) = x^2 y'' + (x+x^2) y' - y = 0$

if $x^2 = 0 \Rightarrow x = 0$

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with

$(x-x_0)^2y'' + (x-x_0)a(x)y' + b(x)y = 0$

(ie) $x^2y'' + x\,a(x)y' + b(x)y = 0$ (11)

$L(y)=0 \Rightarrow x^2y'' + x(1+x)y' - y = 0$

$a(x) = 1+x$, $b(x) = -1$ are analytic at 0

$x=0$ is a regular singular point of $L(y)=0$

13. Compute the indical polynomial and their root for the following equation. 14

$4x^2y'' + (4x^4 - 5x)y' + (x^2+2)y = 0$

solution.

Let $L(y) = 4x^2y'' + (4x^4-5x)y' + (x^2+2)y \rightarrow ①$

$4x^2 = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with

$(x-x_0)^2y'' + (x-x_0)a(x)y' + b(x)y = 0$

At $x_0 = 0$

$x^2y'' + x\,a(x)y' + b(x)y = 0$

Equ ① can be written as

$L(y)=0 \Rightarrow 4x^2y'' + (4x^4 - 5x)y' + (x^2+2)y = 0$

$4x^2y'' + x(4x^3-5)y' + (x^2+2)y$

$x^2y'' + \frac{x(4x^3-5)}{4}y' + \frac{(x^2+2)}{4}y = 0$

$a(x) = \frac{(4x^3-5)}{4}$

$b(x) = \frac{x^2+2}{4}$

at $x=0$

$a(x) = \frac{4x^3}{4} - \frac{5}{4} = -\frac{5}{4}$

$b(x) = \frac{x^2}{4} + \frac{2}{4} = 1/2$

The indical polynomial is

$q(r) = r(r-1) + ar + b$

$q(r) = 0 \Rightarrow r(r-1) - \frac{5}{4}r + 1/2 = 0$

$\Rightarrow r^2 - r - \frac{5}{4}r + 1/2 = 0$

$\Rightarrow r^2 - (1+\frac{5}{4})r + 1/2 = 0$

$\Rightarrow r^2 - \frac{9}{4}r + 1/2 = 0$

(12)

$(\times)4 \quad \Rightarrow 4r^2 - 9r + 2 = 0$

$(r-1/4)(r-2)=0$

$1/4 \mid 2 \qquad \frac{1}{4}+2 = 9/4$

$r = 2, 1/4$

Hence the indicial polynomial roots are 2 and 1/4

14 Compute the indicial polynomial and their roots for the following equation :-

$$x^2y'' + (x-3x^2)y' + e^x y = 0$$

solution: Let $L(y) = x^2y'' + (x-3x^2)y' + e^x y$

$x^2 = 0 \Rightarrow x = 0$

$x=0$ is singular point $L(y)=0$

comparing the given equation with

$$(x-x_0)^2y'' + (x-x_0)\,a(x)y' + b(x)y = 0$$

at $x_0 = 0$ $\quad x^2y'' + x\,a(x)y' + b(x)y = 0$

$L(y)=0 \Rightarrow x^2y'' + x(1-3x)y' + e^x y = 0$

$a(x) = 1-3x \quad ; \; b(x) = e^x$

At $x=0$, $a=1, b=1$

The indicial polynomial

$q(r) = r(r-1) + ar + b$

$= r^2 - r + r + 1$

$q(r) = 0 ; \; r^2 + 1 = 0$

$r = \pm i$

Hence the indicial polynomial roots are i and $-i$

15) Find the singular point of the following equation and determine these which are regular singular point

$$x^2y'' - 5y' + 3x^2y = 0$$

solution: Let $L(y) = x^2y'' - 5y' + 3x^2y$

if $x^2 = 0 \Rightarrow x = 0$

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with

$$(x-x_0)^2y'' + (x-x_0)\,a(x)y' + b(x)y = 0$$

At $x_0=0$

$$x^2y''+x\,a(x)y'+b(x)y=0$$

$$L(y)=0 \Rightarrow x^2y''-5y'+3x^2y=0$$

since the coefficient -5 at y' is not of the form $x\,a(x)$ where $a(x)$ is analytic at 0

$x=0$ is not a regular singular point of $L(y)=0$

16 Find the singular point of the following equation and determine these which are regular singular point

$xy''+4y=0$

Solution:- Let $L(y)=xy''+4y=0$ 13

if $x=0$

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with

$$(x-x_0)^2y''+(x-x_0)\,a(x)y'+b(x)y=0$$

At $x_0=0$

$$x^2y''+x\,a(x)y'+b(x)y=0$$

$$L(y)=0 \Rightarrow xy''+0y'+4y=0$$

$$x(x) \Rightarrow x^2y''+x0y'+4xy=0$$

$a(x)=0$, $b(x)=4x$ are analytic at 0

$\therefore x=0$ is a regular singular point of $L(y)=0$

17. Find the singular point of the following equation and determine these which are regular singular point:

Solution:- Let $L(y)=0$

$$(1-x^2)y''-2xy'+2y=0$$

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$$1-x^2=0$$

$$x^2=1$$

$$x=\pm 1$$

$x=+1,-1$ are singular point of $L(y)=0$

At $x=1$

comparing the given equation with

$$(x-x_0)^2 y'' + (x-x_0)\, a(x) y' + b(x) y = 0$$

$$(x-1)^2 y'' + (x-1)\, a(x) y' + b(x) y = 0$$

$$L(y) = 0$$

$$(1-x^2) y'' - 2xy' + 2y = 0 \qquad (14)$$

$$(1-x)(1+x) y'' - 2xy' + 2y = 0 \qquad \div (1+x)$$

$$(1-x) y'' - \frac{2xy'}{(1+x)} + \frac{2y}{(1+x)} = 0$$

$$\Rightarrow (1-x)^2 y'' - \frac{(1-x) 2xy'}{(1+x)} + \frac{2y(1-x)}{(1+x)} = 0 \qquad \times (1-x)$$

$$\Rightarrow (1-x^2) y'' + \frac{(x-1) 2xy'}{(1+x)} + \frac{2y(1-x)}{(1+x)} = 0$$

$a(x) = \dfrac{2x}{1+x}$; $b(x) = \dfrac{2(1-x)}{(1+x)}$ are analytic at $x_0 = 1$ is a regular singular point of $L(y) = 0$

at $x_0 = -1$ comparing the given equation with -

$$\Rightarrow (x+1)^2 y'' + (x+1) y' a(x) + b(x) y = 0$$

$$L(y) = 0$$

$$\Rightarrow (1-x^2) y'' - 2xy' + 2y = 0$$

$$\Rightarrow (1-x)(1+x) y'' - 2xy' + 2y = 0$$

$$(1+x) y'' - \frac{2xy'}{(1-x)} + \frac{2y}{(1-x)} = 0 \qquad \times (1+x)$$

$$\Rightarrow (1+x)^2 y'' - \frac{(1+x) 2xy'}{(1-x)} + \frac{2y(1+x)}{(1-x)} = 0$$

$a(x) = \dfrac{-2x}{1-x}$; $b(x) = \dfrac{2(1+x)}{(1-x)}$ are analytic at -1

$x_0 = -1$ is a regular singular point of $L(y) = 0$

(18) $3x^2 y'' + x^6 y' + 2xy = 0$

solution :-

let $L(y) = 3x^2y'' + x^6y' + 2xy \rightarrow$ (I)

if $3x^2=0 \Rightarrow x^2=0 \Rightarrow x=0$

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with (15)

$$(x-x_0)^2y'' + x\,a(x)y' + b(x)y=0$$

at $x_0=0$ $\quad x^2y''+x\,a(x)y'+b(x)y=0 \rightarrow$ (I)

(1) $L(y)=0 \Rightarrow 3x^2y''+x^6y'+2xy=0$

$3x^2y''+x(x^5)y'+2xy=0 \div 3$

$\div 3)$

$$\Rightarrow x^2y''+\frac{x^6}{3}y'+\frac{2xy}{3}=0$$

comparing (1) & (2)

$$x^2y''+x\left(\frac{x^5}{3}\right)y'+\frac{2x}{3}y=0 \rightarrow (2)$$

$a(x)=\frac{x^5}{3}$; $b(x)=\frac{2x}{3}$ are analytic at $x=0$

$x=0$ is a regular singular point of $L(y)=0$

19. $(x^2+x-2)^2y''+3(x+2)y'+(x-1)y=0$ NOV-15

solution:-

let $L(y)=(x^2+x-2)^2y''+3(x+2)y'+(x-1)y$

if $(x^2+x-2)^2=0 \Rightarrow x^2+x-2=0$

$(x+2)(x-1)=0$

At $x_0=-2$ $\qquad x=-2,1$

comparing the given equation with

$$(x-x_0)^2y''+(x-x_0)a(x)y'+b(x)y=0$$

$$(x+2)^2y''+(x+2)a(x)y'+b(x)y=0$$

let $L(y)=(x^2+x-x)^2+3(x+2)y'+(x-1)y=0$

$$(x+2)^2(x-1)^2+3(x+2)y'+(x-1)y=0$$

$$(x+2)^2y''+\frac{3(x+2)}{(x-1)^2}y'+\frac{(x-1)}{(x-1)^2}y=0$$

20

$$(x+2)^2 + \frac{3(x+2)}{(x-1)^2}y' + \frac{1}{(x-1)}y = 0$$ 16

$a(x) = \frac{3}{(x-1)^2}$, $b(x) = \frac{1}{(x-1)}$ are analytic at -2

$x_0 = 0$ is a regular singular point of $L(y)=0$ at $x_0 = 1$

Comparing the given equation with

$$(x-x_0)^2 y'' + (x-x_0)\, a(x)\, y' + b(x) y = 0$$

$$(x-1)^2 y'' + (x-1)\, a(x)\, y' + b(x) y = 0$$

Let $L(y) = (x^2+x-2)^2 y'' + 3(x+2)y' + (x-1)y = 0$

$$(x+2)^2(x-1)^2 + 3(x+2)y' + (x-1)y = 0$$

$$(x-1)^2 y'' + \frac{3(x+2)}{(x+2)^2}y' + \frac{(x-1)}{(x+2)^2}y = 0$$

$$(x-1)^2 y'' + \frac{3}{(x+2)}y' + \frac{(x-1)}{(x+2)^2}y = 0$$

since the co-efficient $\frac{3}{x+2}$ of y' is not of the form $x\,a(x)$ where $a(x)$ is analytic at 1.

$\therefore x_0 = 1$ is not regular singular point of $L(y)=0$ y

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20. $xy'' + (1-x)y' + \alpha y = 0$ Where α is constant is called Legendre equation

a) show that this equation has a regular singular point at $x=0$

b) Compute the indical polynomial and its roots

c) Find a solution ϕ of the form $\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$

Solution :-

a) $L(y) = xy'' + (1-x)y' + \alpha y = 0$ — ①

if $x = 0$

$x = 0$ is singular point of $L(y) = 0$

Comparing the given equation with

$$(x-x_0)^2 y'' + (x-x_0)\,a(x)y' + b(x)y = 0$$

At $x_0 = 0$

$$x^2 y'' + x\,a(x)y' + b(x)y = 0$$

$L(y) = 0 \Rightarrow xy'' + (1-x)y' + \alpha y = 0$

x multiply $\Rightarrow x^2 y'' + x(1-x)y' + x\alpha y = 0$

$a(x) = 1-x$, $b(x) = \alpha x$ are analytic at 0

$x = 0$ is a regular singular point of $L(y) = 0$

At $x = 0$

$$a = 1, \; b = 0$$

b) $q(r) = r(r-1) + ar + b$

$= r^2 - r + ar + b$

$= r^2 - r + r$

$q(r) = 0 \Rightarrow r^2 = 0$

$r_1 = 0$; $r_2 = 0$

The indicial polynomial roots are 0 and 0

c) A solution of $L(y) = 0$ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} c_k \cdot x^{k+r}$$

$$\phi'(x) = \sum_{k=0}^{\infty} (k+r)\,c_k\,x^{k+r-1}$$

$$\phi''(x) = \sum_{k=0}^{\infty} (k+r)(k+r-1)\,c_k \cdot x^{k+r-2}$$

since $L(\phi)=0$

$$x\phi'' + (1-x)\phi' + \alpha\phi = 0$$

$$x^2\phi'' + x(1-x)\phi' + \alpha x\phi = 0$$ (18)

$$x^2 \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) \cdot x^{k+r-2} + (x-x^2) \cdot \sum_{k=0}^{\infty} c_k (k+r) \, x^{k+r-1} + \alpha x \sum_{k=0}^{\infty} c_k \cdot x^{k+r} = 0$$

$$\sum_{k=0}^{\infty} c_k (k+r)(k+r-1)\, x^{k+r} + \sum_{k=0}^{\infty} c_k (k+r)\, x^{k+r} - \sum_{k=0}^{\infty} c_k (k+r)\, x^{k+r+1} + \alpha \sum_{k=0}^{\infty} c_k \cdot x^{k+r+1} = 0$$

$$c_0\, r(r-1)x^r + \sum_{k=1}^{\infty} c_k (k+r)(k+r-1) \cdot x^{k+r} + c_0\, r x^r + \sum_{k=1}^{\infty} c_k (k+r) \cdot x^{k+r} - \sum_{k=1}^{\infty} c_{k-1} (k+r-1)\, x^{k+r} + \alpha \sum_{k=\phi}^{\infty} c_{k-1}\, x^{k+r} = 0$$

$$c_0\, x^r (r(r-1)+r) + x^r \sum_{k=1}^{\infty} \left\{ [(k+r)(k+r-1)+(k+r)]\, c_k + [-(k+r-1)+\alpha]\, c_{k-1} \right\} x^k = 0$$

$$c_0\, x^r (r^2 - r + r) + x^r \sum_{k=1}^{\infty} [(k+r)^2 c_k + (\alpha - k - r + 1) c_{k-1}] \cdot x^k = 0$$

Let $c_0 = 1$

$$(k+r)^2 c_k + (\alpha - k - r + 1)\, c_{k-1} = 0$$

$$c_k = \frac{c_{k-1}(-\alpha + k + r - 1)}{(k+r)^2}, \quad k = 1, 2, \ldots$$

$$c_{k-1} = \frac{c_{k-2}(-\alpha+k+r-2)}{(k+r-1)^2}$$

$$\vdots$$

$$c_1 = \frac{c_0(-\alpha+r)}{(r+1)^2}$$

(19)

$$c_k = \frac{(-\alpha+k+r-1)\cdots(-\alpha+r)c_0}{(k+r)^2\cdots(r+1)^2} \quad k=1,2,\ldots$$

$$\phi(x) = c_0 x^r + x^r \sum_{k=1}^{\infty} \frac{(-\alpha+k+r-1)\cdots(-\alpha+r)c_0}{(k+r)^2\cdots(r+1)^2} x^k$$

if $r=0$ and $c_0=1$

$$\phi(x) = 1 + \sum_{k=1}^{\infty} \frac{(-\alpha+k-1)\cdots(-\alpha+0)}{(k^2)\cdots(1)^2} x^k$$

Hence

$$\phi(x) = 1 + \sum_{k=1}^{\infty} \frac{(-\alpha+k-1)\cdots(-\alpha+0)}{(k!)^2} x^k$$

21. Find the solution of the equation $x^2y'' + \frac{3}{2}xy' + xy = 0$

Solution: Let $L(y) = x^2y'' + \frac{3}{2}xy' + xy = 0$

$\phi(x)$ is the solution of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} c_k x^{r+k}$$

$$\phi'(x) = \sum_{k=0}^{\infty} c_k (r+k) x^{r+k-1}$$

$$\phi''(x) = \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k-2}$$

Since $\phi(x)$ be the solution of $L(y)=0$

$$x^2\phi'' + \tfrac{3}{2}x\phi' + x\phi = 0$$

$$\Rightarrow x^2 \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k-2} + \frac{3}{2} x \sum_{k=0}^{\infty} c_k (r+k) x^{r+k-1}$$

$$+ x \sum_{k=0}^{\infty} c_k x^{r+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} c_k (r+k)(r+k-1) x^{r+k} + 3/2 \sum_{k=0}^{\infty} c_k (r+k) x^{r+k} +$$

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$$\sum_{k=0}^{\infty} c_k x^{r+k+1} = 0$$

$$\Rightarrow c_0 (r)(r-1)x^r + \sum_{k=1}^{\infty} c_k (r+k)(r+k-1) x^{r+k} + 3/2\, c_0 r (x^r)$$

$$+ 3/2 \sum_{k=1}^{\infty} c_k (r+k) x^{r+k} + \sum_{k=1}^{\infty} c_{k-1} x^{r+k} = 0$$

$$\Rightarrow [r(r-1) + 3/2\, r] x^r c_0 + \sum_{k=1}^{\infty} x^r \{ c_k (r+k)(r+k-1) + 3/2$$

$$(r+k) + c_{k-1} \} x^k = 0 \quad \longrightarrow ①$$

Let $q(r) = r(r-1) + 3/2\, r$

$$= r^2 - r + 3/2\, r$$

$$= r^2 + r/2 = r(r + 1/2)$$

similarly, $q(r+k) = (r+k)(r+k-1) + 3/2\,(r+k)$

$$= (r+k)(r+k-1+3/2)$$

$$= (r+k)(r+k+1/2)$$

Equation ① becomes

$$q(r)\, x^r c_0 + \left(x^r \sum_{k=0}^{\infty} q(r+k) c_k + c_{k-1}\right) x^k = 0$$

Let $q(r) = 0$

$$q(r+k) c_k + c_{k-1} = 0$$

which is an indical polynomial for given equation

$$q(r) = 0 \Rightarrow r(r + 1/2) = 0$$

$$r_1 = 0, \quad r_2 = -1/2$$

$$q(r+k) c_k + c_{k-1} = 0 \Rightarrow c_k = \frac{-c_{k-1}}{q(r+k)} \rightarrow ②$$

$$c_{k-1} = \frac{-c_{k-2}}{q(r+k-1)}$$

$$c_{k-2} = \frac{-c_{k-3}}{q(r+k-2)}$$

$$c_1 = \frac{-c_0}{q(r+k)}$$

sub in ①

$$c_k = \frac{(-1)^k c_0}{q(r+k)\, q(r+k-1) \cdots q(r+1)}$$

(21)

$$\phi(x) = c_0 x^r + \sum_{k=1}^{\infty} \frac{(-1)^k c_0 \, x^{k+r}}{q(r+k)\, q(r+k-1) \cdots q(r+1)}$$

Case (i)

At $r_1 = 0$, $c_0 = 1$

$$\phi(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k)\, q(k-1) \cdots q(1)}$$

Case (ii)

At $r_2 = -1/2$, $c_0 = 1$

$$\phi(x) = x^{-1/2} + \sum_{k=1}^{\infty} \frac{(-1)^k x^{k-1/2}}{q(k-1/2)\, q(k-3/2) \cdots q(1/2)}$$

Theorem : 3.2

Consider the equation $x^2 y'' + a(x) y' + b(x) y = 0$ where a, b have power series expansions which are convergent for $|x| < r_0$, $r_0 > 0$

Let $r_1 r_2$ ($\operatorname{Re} r_1 \geq \operatorname{Re} r_2$) be the roots of the indical polynomial

$$q(r) = r(r-1) + a(0) r + b(0)$$

If $r_1 = r_2$ there are two linearly independent solutions ϕ_1, ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_1(x) = |x|^r \sigma_1(x)$$

$$\phi_2(x) = |x|^{r_1+1} \sigma_2(x) + (\log |x|)\, \phi_1(x)$$

Where σ_1, σ_2 have power series expansion which independent solutions $\phi_1 \phi_2$ for $0 < |x| < r_0$ of the form $\phi_1(x) = |x|^{r_1} \sigma_1(x)$

$$\phi_2(x) = |x|^{r_2} \sigma_2(x) + c(\log |x|)$$

where σ_1 σ_2 have power series expansions which are convergent for $|x|<r_0$, $\sigma_1(0)\neq 0$, $\sigma_2(0)\neq 0$ and C is constant. it may happen that $C=0$

22. consider the equation $x^2y''+xy'+(x^2-\alpha^2)y=0$ where α is constant

a) compute the Indicial polynomial and its root

b) Discuss the nature of the solution near the origin consider all cases carefully do not compute the solutions

NOV-18

a) $L(y)=x^2y''+xy'+(x^2-\alpha^2)y=0$ NOV-17

if $x^2=0$ $\Rightarrow x=0$ (22) Nov-15

$x=0$ is a singular point of $L(y)=0$

comparing the given equation with

$$(x-x_0)^2y''+(x-x_0)a(x)y'+b(x)y=0$$

$$x^2y''+x\,a(x)y'+b(x)y=0$$

$a(x)=1$, $b(x)=x^2-\alpha^2$ are analytic at 0

$x=0$ is a regular singular point of $L(y)=0$

at $x=0$ $\Rightarrow a=1$, $b=-\alpha^2$

The indicial Polynomial

$$q(r)=r(r-1)+ar+b$$

$$=r^2-r+ar+b$$

$$q(r)=0\Rightarrow r^2-\alpha^2=0$$

$r=\pm\alpha$. The roots of indicial polynomial are $\alpha,-\alpha$

b) If $\alpha=0$

There are two solutions of the term

$$\phi_1(x)=\sigma_1(x)$$

$$\phi_2(x)=x\sigma_2(x)+C(\log|x|)\phi_1(x)$$

Where σ_1, σ_2 are power series convergent for $|x|<\alpha$

~~if $\alpha\neq 0$~~ if $\alpha>0$

2α is not a +ve integer.

Two solutions are of the form

$$\phi_1(x) = |x|^{\alpha}\sigma_1(x)$$
$$\phi_2(x) = |x|^{-\alpha}\sigma_2(x)$$

Where $\sigma_1 \sigma_2$ are two power series convergent for $|x| < \alpha$

If 2α is a +ve integer two solutions have of the form

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$$\phi_1(x) = |x|^{\alpha}\sigma_1(x)$$
$$\phi_2(x) = |x|^{-\alpha}\sigma_2(x) + c(\log|x|)\phi_1(x)$$

Where $\sigma_1 \sigma_2$ are two power series convergent for $|x| < \alpha$

---

23 a) show that -1 and 1 are singular points for the legendre equations

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

b) Find the indical Polynomial and its roots Corresponding to the Point $x=1$

Proof:- $L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$$1 - x^2 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

$x = 1, -1$ are singular points of $L(y) = 0$

At $x_0 = 1$

comparing the given equation with

$$(x-x_0)^2 y'' + (x-x_0)\, a(x) y' + b(x) y = 0$$
$$(x-1)^2 y'' + (x-1)\, a(x) y' + b(x) y = 0$$
$$L(y) = 0 \Rightarrow (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$
$$= (1+x)(1-x)y'' - 2xy' + \alpha(\alpha+1)y = 0 \div$$

$(1+x)$

$$= (1-x)y'' - \frac{2xy'}{(1+x)} + \frac{\alpha(\alpha+1)}{(1+x)}y = 0 \quad (\times)(1-x)$$

$$\Rightarrow (1+x^2)y'' - \frac{2x(1+x)}{(1-x)}y' + \frac{\alpha(\alpha+1)(1+x)}{(1-x)}y = 0$$

$$\Rightarrow (x+1)^2 y'' - \frac{2x(1+x)}{(1-x)}y' + \frac{\alpha(\alpha+1)(1+x)}{(1-x)}y = 0$$

$$a(x) = \frac{-2x}{1-x}y'' \; ; \; b(x) = \frac{\alpha(\alpha+1)(1+x)}{(1-x)}$$

are analytic at -1

$\therefore x = -1$ is regular singular Point

b) Proof: $x = 1$

$$a(x) = \frac{2x}{1+x} = \frac{2}{2} = 1$$

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$$b(x) = \frac{\alpha(\alpha+1)(1-x)}{1+x}$$

$$b(x) = 0$$

The indical polynomial

$$q(r) = r(r-1) + ar + b$$
$$= r^2 - r + r + 0$$
$$= r^2 + 0$$
$$r = 0$$

Apr-19, Nov-18

Derive the solution of Bessel equation

If α is constant $Re\,\alpha = 0$ the Bessel equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

This has the form

$$x^2y'' + x \cdot a(x)y' + b(x)y = 0$$

$$a(x) = 1 \quad , \quad b(x) = x^2 - \alpha^2$$

since a, b are analytic at $x = 0$

The Bessel equation has the origin as a regular point The indical polynomial q is given by

$$= r^2 - \cancel{r} + \cancel{r} - \alpha^2 \quad q(r) = r(r-1) + r - \alpha^2 = r^2 - \alpha^2$$

Whose two roots $r_1 r_2$ are $r_1 = \alpha$, $r_2 = -\alpha$

We shall construct the solutions for $x>0$

Let us consider the case for $\alpha=0$

Case (i) $\Rightarrow r_1=0$ and $r_2=0$

$\therefore$ The solutions ϕ_1, ϕ_2 of the form (25)

$$\phi_1(x)=\sigma_1(x)$$

$$\phi_2(x)=\sigma_2(x)+(\log x)\,\phi_1(x)$$

where $\sigma_1\,\sigma_2$ have the power series expansion which converges for all finite x

To compute $\sigma_1\,\sigma_2$ $\quad x^2y''+xy'+(x^2-\alpha^2)y=0$

$\alpha=0$

Let $L(y)=x^2y''+xy'+x^2y$

Let $y=\sigma_1(x)$

suppose $\phi_1(x)=\sigma_1(x)=\sum_{k=0}^{\infty} c_k\cdot x^k \quad (\because c_0\neq 0)$

we find

$$\sigma_1'(x)=\sum_{k=1}^{\infty} k\,c_k\cdot x^{k-1}$$

$$\sigma_1''(x)=\sum_{k=2}^{\infty} k(k-1)\cdot c_k\,x^{k-2}$$

since $\sigma_1(x)$ is a solution $L(y)=0$

$$x^2\sigma_1''(x)+x\,\sigma_1'(x)+(x^2-\alpha^2)\sigma_1(x)=0$$

$$x^2\sum_{k=2}^{\infty} k(k-1)c_k\,x^{k-2}+x\sum_{k=1}^{\infty} k\,c_k\,x^{k-1}$$

$$+(x^2)\sum_{k=0}^{\infty} c_k x^k=0$$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k+\sum_{k=1}^{\infty} k\,c_k\,x^k+\sum_{k=0}^{\infty} c_k\,x^{k+2}=0$$

$$\sum_{k=2}^{\infty} k(k-1)\,c_k x^k+c_1x+\sum_{k=2}^{\infty} k c_k x^k+\sum_{k=2}^{\infty} c_{k-2}\,x^k=0$$

Thus

$$L(\sigma_1(x))=c_1x+\sum_{k=2}^{\infty}\left[(k(k-1)+k)\,c_k+c_{k-2}\right]x^k=0$$

$\phi_1(x)=c_0x^{\alpha}+c_1x^{1+\alpha}+c_2x^{2+\alpha}+c_3x^{3+\alpha}$

$c_1 = 0$

$[k(k-1)+k]\, c_k + c_{k-2} = 0 \quad , k = 1,2,3,\ldots$

$\Rightarrow \quad (k^2 - k + k)\, c_k + c_{k-2} = 0$

$$c_k = \frac{-c_{k-2}}{k^2}$$

Let $c_0 = 1$ implies (26)

$$c_2 = -\frac{1}{2^2}, \quad c_4 = \frac{-c_2}{4^2} = \frac{1}{2^2 \cdot 4^2}$$

and general

$$c_{2m} = \frac{(-1)^m}{2^2 \cdot 4^2 \cdots (2m)^2} = \frac{(-1)^m}{2^{2m} (m!)^2} \quad (m = 1, 2, \ldots$$

since $c_1 = 0$ we have

$$c_3 = c_5 = \cdots = 0$$

Thus σ_1 contains only even power of x and we obtain

$$\sigma_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2}$$

Where as usual $0! = 1 \cdot 2^0 = 1$

The function defined by this series is called the Bessel function of zero order of the first kind and is denoted by J_0

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

which converges for all finite x]

Case (ii) Let $\phi_1 = J_0$

$$\phi_2(x) = \sum_{k=0}^{\infty} c_k x^k + (\log x)\, \phi_1(x), \quad (c_0 = 0)$$

$$\phi_2'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1} + \frac{\phi_1(x)}{x} + \log x\, \phi_1'(x)$$

$$\phi_2''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \frac{\phi_1(x)}{x^2} + \frac{2}{x}\phi_1'(x) + (\log x)\, \phi_1''(x)$$

Thus,

$$L(\phi_2(x)) = x^2\phi_2''(x) + x\phi_2'(x) + x^2\phi_2(x)$$

$$= x^2\left\{\sum_{k=2}^{\infty} k(k-1)\, c_k\, x^{k-2} - \frac{\phi_1(x)}{x} + \frac{2}{x}\phi_1'(x) + (\log x)\,\phi_1''(x)\right\}$$

$$+ x\left\{\sum_{k=1}^{\infty} k c_k x^{k-1} + \frac{\phi_1(x)}{x} + (\log x)\phi_1'(x)\right\}$$

$$+ x^2\left\{\sum_{k=0}^{\infty} c_k x^k + (\log x)\,\phi_1(x)\right\}$$

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$$= \sum_{k=2}^{\infty} k(k-1)\, c_k x^k - \phi_1(x) + 2x\phi_1'(x) + (x^2\log x)\,\phi_1''(x) + \sum_{k=1}^{\infty} k c_k x^k + \phi_1(x) + (x\log x)\phi_1'(x)$$

$$+ \sum_{k=0}^{\infty} c_k x^{k+2} + (x^2\log x)\,\phi_1(x).$$

$$= 2c_2x^2 + \sum_{k=3}^{\infty} k(k-1)\, c_k x^k + 2x\phi_1'(x) + \log x$$

$$\left\{x^2\phi_1''(x) + x\phi_1'(x) + x^2\phi_1(x)\right\} - \phi_1(x) + c_1x +$$

$$2c_2x^2 + \sum_{k=3}^{\infty} k c_k x^k + \phi_1(x) + \sum_{k=3}^{\infty} \cancel{c_{k-2}}\, x^k + c_0x^2$$

c_{k-2}

$$= c_1x + 2^2c_2x^2 + 2x\phi_1'(x) + \log x\,(L(\phi_1(x)) +$$

$$\sum_{k=3}^{\infty}\left\{[k(k-1)+k]\,c_k + c_{k-2}\right\}x^k + c_0x^2$$

$$\sum_{k=3}(k^2c_k + (k-2)\,x^k$$

$$L(\phi_2(x)) = c_1x + 2^2c_2x^2 + c_0x^2 + 2x\phi_1'(x) + \sum_{k=3}^{\infty}(k^2 c_k + c_{k-2})$$

$$L(\phi_2(x)) = 0$$

$$\Rightarrow c_1x + 2^2c_2x^2 + \sum_{k=3}^{\infty}(k^2c_k + c_{k-2})x^k$$

$$= -2x \sum_{m=0}^{\infty} \frac{(-1)^m \, 2m \cdot x^{2m-1}}{(m!)^2 \cdot 2^{2m}}$$

$$= -2 \sum_{m=0}^{\infty} \frac{(-1)^m \cdot 2m}{(m!)^2 \cdot 2^{2m}} x^{2m}$$

Equating the coefficients of power of x on both sides

$$c_1 = 0, \quad 2^2 c_2 = \frac{-2(-1)\,2\cdot 1}{(1!)^2 (2)^2} \Rightarrow c_2 = \frac{1}{2^2}$$

$$3^2 c_3 + c_1 = 0$$

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$$c_3 = \frac{-c_1}{3^2} = 0$$

$$(2m)^2 c_{2m} + c_{2m-2} = \frac{(-1)^m \cdot 2m}{(m!)^2 \cdot 2^{2m}} \times (-2)$$

$$= \frac{(-1)^{m+1} \cdot m}{(m!)^2 \, 2^{2m-2}} \qquad = \frac{-2(1)(1)}{2(1\,2)}$$

we have, $c_2 = \frac{1}{2^2}$

$$c_4 = \frac{1}{4^2}\left(\frac{-1}{2^2} - \frac{1}{2\cdot 2^2}\right) = \frac{-1}{2^2 \cdot 4^2}(1 + 1/2)$$

$$c_6 = \frac{1}{6^2}\left[\frac{1}{2^2 4^2}(1 + 1/2) + \frac{1}{2^2 \cdot 4^2}(1/3)\right]$$

$$= \frac{1}{2^2 \cdot 4^2 \cdot 6^2}(1 + 1/2 + 1/3 + \cdots)$$

$$c_{2m} = \frac{(-1)^{m-1}}{2^m (m!)^2}(1 + 1/2 + 1/3 + \cdots + 1/m) \quad (m = 1, 2, \ldots)$$

The solution thus determined is called a Bessel function of zero order r of the second and is denoted by K_0

Hence $$K_0(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}(1 + 1/2 + \cdots + 1/m) \cdot [x/2]^{2m}$$

$+ (\log x) J_0(x)$ which is convergent for all finite x

Formula :- 1.

A solution of the Bessel equation of order α which is denoted by J_α

$$\boxed{J_\alpha(x) = (x/2)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(x/2\right)^{2m}}$$

where Γ is the gamma function defined by

$$\Gamma(z) = \int_{0}^{\infty} e^{-x} x^{z-1}\,dx \quad (\text{Re } z \geq 0)$$

Also

$$\boxed{\Gamma(z+1) = z\,\Gamma(z)}$$

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$$\Gamma(n+1) = n! \text{ and } \Gamma(1) = 1$$

Note :- (1)

The Formula for J_α reduces to J where $\alpha = 0$ since $\Gamma(m+1) = m!$

Note :- (2)

$$J_{-\alpha}(x) = (x/2)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-\alpha+1)} \left(x/2\right)^{2m}$$

Formula : 2

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-1} \sum_{j=0}^{n-1} \frac{n-j-1}{j!} \left(x/2\right)^{2} - \frac{1}{2} \cdot \frac{1}{n!}$$

$$\left(1 + 1/2 + \cdots + 1/m\right)\left(x/2\right)^n - 1/2\left(x/2\right)^n$$

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(m+n)!} \left[\left(1+1/2+\cdots+1/m\right)\right] + \left(1 + 1/2 + \cdots + \frac{1}{m+n}\right)\Bigg]$$

$$\left(x/2\right)^{2m} + (\log x)\, J_n(x)$$

This formulae reduces to the one for $K_0(x)$ when $n=0$ provided we interpret this first two sums on the right as zero in this case the function K_n is called the

Bessel function of order n of the second kind

1. Show that $x^{1/2} J_{1/2} x = \frac{\sqrt{2}}{\Gamma(1/2)} \sin x$ Apr-19

(30)

we know that

Apr-19 $J_\alpha(x) = (x/2)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} (x/2)^{2m}$

$J_{1/2}(x) = (x/2)^{1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1/2+1)} (x/2)^{2m}$

$x^{1/2} J_{1/2}(x) = \frac{x^{1/2} \cdot x^{1/2}}{2^{1/2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1/2+1)} (x/2)^{2m}$

$= \frac{x}{\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+1/2+1)} (x/2)^{2m}$

(18)

$= \frac{1}{\sqrt{2}} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{m!\,\Gamma(m+1/2+1) \cdot 2^{2m}}$

$= \frac{1}{\sqrt{2}} \left\{ \frac{1}{\Gamma(1/2+1) \cdot 1} x - \frac{x^3}{1!\,\Gamma(1+1/2+1) \cdot 2^2} + \frac{x^5}{2!\,\Gamma(2+1/2+1) \cdot 2^4} + \cdots \right\}$

$= \frac{1}{\sqrt{2}} \left\{ \frac{x}{\Gamma(1/2+1)} - \frac{x^3}{2^2\,\Gamma(3/2+1)} + \frac{x^5}{2!\,2^4\,\Gamma(5/2+1)} - \cdots \right\} \rightarrow (1)$

We know that

$\Gamma(x+1) = z\Gamma(z)$

$\Gamma(1/2+1) = 1/2\,\Gamma(1/2)$

$\therefore \Gamma(3/2+1) = 3/2\,\Gamma(3/2)$

$= 3/2 \cdot 1/2\,\Gamma(1/2)$

$\Gamma(5/2+1) = 5/2\,\Gamma(5/2)$

$= 5/2 \cdot 3/2 \cdot 1/2\,\Gamma(1/2)$

$\frac{x^5}{2!\,2^4\,5/2 \cdot 3/2\,\Gamma(1/2)}$

Substituting these value in (1) we get

$x^{1/2} J_{1/2}(x) = \frac{1}{\sqrt{2}} \left\{ \frac{x}{1/2\,\Gamma(1/2)} - \frac{x^3}{2^2 \cdot 3/2 \cdot 1/2\,\Gamma(1/2)} + \frac{x^5}{2!\,2^4\,5/2 \cdot 3/2\,\Gamma(1/2)} \right\}$

$$= \frac{1}{\sqrt{2}\cdot \frac{1}{2}\,\Gamma(1/2)} \left\{ x - \frac{x^3}{2^2\cdot 3/2} + \frac{x^5}{2^4\, 2!\, 5/2\cdot 3/2} \cdots\cdots \right\}$$

$$= \frac{\sqrt{2}}{\Gamma(1/2)} \left\{ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right\}$$

$$= \frac{\sqrt{2}}{\Gamma(1/2)} \sin x \qquad \Gamma(1/2) = \sqrt{\pi}$$

$$= x^{1/2} J_{1/2}(x) = \frac{\sqrt{2}}{\Gamma(1/2)} \sin x$$

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2 Show that $x^{1/2} J_{-1/2}(x) = \frac{\sqrt{2}}{\Gamma(1/2)} \cos x$ NOV-18 Ap-18

solution:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

$$J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-1/2+1)} \left(\frac{x}{2}\right)^{2m}$$

$$x^{1/2} J_{-1/2}(x) = \frac{x^{1/2}\, x^{-1/2}}{2^{-1/2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-1/2+1)} \left(\frac{x}{2}\right)^{2m}$$

$$= \sqrt{2} \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m-1/2+1)} \left(\frac{x}{2}\right)^{2m}$$

$$= \sqrt{2} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m}}{m!\,\Gamma(m-1/2+1)\cdot 2^{2m}}$$

$$= \frac{1}{\sqrt{2}} \left\{ \frac{1}{\Gamma(-1/2+1)} - \frac{x^2}{1\,\Gamma(1-1/2+1)\,2^2} + \frac{x^4}{2!\,\Gamma(2-1/2+1)\cdot 2^4} \cdots \right\}$$

$$x^{1/2} J_{-1/2}(x) = \sqrt{2} \left\{ \frac{1}{\Gamma(0-\frac{1}{2}+1)} - \frac{x^2}{2^2\,\Gamma(1/2+1)} + \frac{x^4}{2^4\, 2!\,\Gamma(3/2+1)} \cdots \right.$$

We know that

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(1/2+1) = 1/2\;\Gamma(1/2)$$

$$\Gamma(3/2+1) = 3/2\cdot\Gamma(3/2)$$

$= 3/2 \cdot 1/2\, \Gamma(1/2)$

substitute these values in ① we get

$$x^{1/2} J_{-1/2}(x) = \sqrt{2} \left\{ \frac{1}{\Gamma(1/2)} - \frac{x^2}{2^2 2!\, \Gamma(1/2)} + \frac{x^4}{2^4 \cdot 2!\, 3/2 \cdot 1/2\, \Gamma(1/2)} - \cdots \right\}$$

$$= \frac{\sqrt{2}}{\Gamma(1/2)} \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right\}$$

cos x

$$x^{-1/2} J_{-1/2}(x) = \frac{\sqrt{2}}{\Gamma(1/2)} \cos x$$

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3) a) show that $J_0'(x) = -J_1(x)$

b) prove that between any two positive zeros of J_0 there is a zero of J_1 NOV-17

solution: a) we know that

$$J_\alpha(x) = (x/2)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+\alpha+1)} (x/2)^{2m}$$

$$J_1(x) = (x/2) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+1+1)} (x/2)^{2m}$$

$$= x/2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+2)} (x/2)^{2m}$$

$$J_0(x) = (x/2)^0 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+0+1)} (x/2)^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, \Gamma(m+1)} (x/2)^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\, m!} (x/2)^{2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x^{2m}}{2^{2m}}\right)$$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \cdot \frac{1}{2^{2m}} \cdot 2m \cdot x^{2m-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{[(m+1)!]^2} \cdot \frac{1}{2^{2(m+1)}} \cdot 2(m+1) \cdot x^{(2(m)+}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^1}{((m+1)(m!))^2} \cdot \frac{1}{2^{2(m+1)}} \cdot 2(m+1) \cdot x^{2m+1}$$

$$= -\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^2 (m!)^2} \cdot \frac{1}{2^{2m} \cdot 2^2} \cdot 2(m+1)\ x^{2m} \cdot x^1$$

$$= -x/2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m!)^2} \left(x/2\right)^{2m}$$

$$= -J_1(x)$$

$$J_0'(x) = -J_1(x)$$

b) The Rolle's theorem state that "Between two consecutive real roots a and b of the equation $f(x)=0$, the derivatives of $f'(x)=0$ has the atleast one real root"

$\therefore$ Between any positive zero of $J_0(x)=0$ has atleast one positive zero

ie) $J_1(x)=0$ has atleast one positive zero

4 Show that (i) $(x^{\alpha} \cdot J_{\alpha})'(x) = x^{\alpha} J_{\alpha-1}(x)$

(ii) $(x^{-\alpha} J_{\alpha})'(x) = -x^{-\alpha} J_{\alpha+1}(x)$

solution: (i) we know that

$$J_{\alpha}(x) = (x/2)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(x/2\right)^{2m}$$

$$x^{\alpha} J_{\alpha}(x) = x^{\alpha} \left(\frac{x^{\alpha}}{2^{\alpha}}\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(x/2\right)^{2m}$$

$$= \frac{x^{2\alpha}}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \frac{x^{2m}}{2^{2m}}$$

$$= \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \cdot \frac{x^{2m+2\alpha}}{2^{2m}}$$

$$= \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+\alpha)\,\Gamma(m+\alpha)} \cdot \frac{x^{2m+2\alpha}}{2^{2m}}$$

$$(x^{\alpha} J_{\alpha})'(x) = 1/2^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(m+\alpha)}{m!\,(m+\alpha)\,\Gamma(m+\alpha)} \cdot \frac{x^{2m+2\alpha-1}}{2^{2m}}$$

$$= \frac{x^{2\alpha-1}}{2^{\alpha-1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha)} \cdot \left(x/2\right)^{2m}$$

$$= \frac{x^{\alpha}\cdot x^{\alpha-1}}{2^{\alpha-1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha)} \left(x/2\right)^{2m}$$ Bessel

$$= x^{\alpha} J_{\alpha-1}(x)$$

$$(x^{\alpha} J_{\alpha})'(x) = x^{\alpha} J_{\alpha-1}(x)$$ 34

(ii) We know that

$$J_\alpha(x) = (x/2)^{\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \cdot \left(x/2\right)^{2m}$$

$$x^{-\alpha} J_\alpha(x) = x^{-\alpha}\left(\frac{x^{\alpha}}{2^{\alpha}}\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(x/2\right)^{2m}$$

$$= \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(x/2\right)^{2m}$$ $x^{2(m+1)-1}$ x^{2m+2-1} $x^{2m}x^{1}$

$$(x^{-\alpha} J_\alpha)'(x) = 1/2^{\alpha} \cdot \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \cdot \frac{2m\cdot x^{2m-1}}{2^{2(m+1)}}$$

$$= \frac{1}{2^{\alpha}} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}\,2(m+1)}{(m+1)\,m!\,\Gamma(m+2+\alpha)} \cdot \frac{x^{2m}x}{2^{2m}\cdot 2^{2}}$$ $2(m+1)$

$$= -x^{-\alpha}\frac{x^{\alpha}}{2^{\alpha}}\left(x/2\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+2+\alpha)} \left(x/2\right)^{2m}$$

$$= -x^{-\alpha}\frac{x^{\alpha+1}}{2^{\alpha+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+2+\alpha)} \cdot \left(x/2\right)^{2m}$$

$$= -x^{-\alpha} J_{\alpha+1}(x)$$

$$-x^{-\alpha} J_\alpha(x) = -x^{-\alpha} J_{\alpha+1}(x)$$

Hence Proved

Show that $J_{\alpha-1}(x) - J_{\alpha+1}(x) = 2J_\alpha'(x)$ and $J_{\alpha-1}(x) + J_{\alpha+1}(x) = 2\alpha x^{-1} J_\alpha(x)$

solution: We know that

$$(x^{\alpha} J_{\alpha})'(x) = x^{\alpha} J_{\alpha-1}(x) \rightarrow ①$$

$$(x^{-\alpha} J_{\alpha})'(x) = -x^{-\alpha} J_{\alpha+1}(x) \rightarrow ②$$

$$① \Rightarrow J_{\alpha-1}(x) = (x^{\alpha} J_{\alpha})'(x) \cdot x^{-\alpha} \rightarrow ③$$

$$② \Rightarrow J_{\alpha+1}(x) = -(x^{-\alpha} J_{\alpha})'(x) \cdot x^{\alpha} \rightarrow ④$$

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subtract ③ and ④

$$J_{\alpha-1}(x) - J_{\alpha+1}(x) = (x^{\alpha} J_{\alpha})'(x) \cdot x^{-\alpha} + (x^{-\alpha} J_{\alpha})'(x) \cdot x^{\alpha}$$

$$\Rightarrow x^{-\alpha}(x^{\alpha} J_{\alpha}'(x) + J_{\alpha}(x) \cdot \alpha x^{\alpha-1}) + x^{\alpha}(x^{-\alpha} J_{\alpha}'(x) - \alpha x^{-\alpha-1} J_{\alpha}(x))$$

$$\Rightarrow x^{-\alpha} x^{\alpha} J_{\alpha}'(x) + x^{-\alpha} J_{\alpha}(x) \cdot \alpha x^{\alpha-1} + x^{\alpha} x^{-\alpha} \cdot J_{\alpha}'(x) - x^{\alpha} \cdot \alpha x^{-\alpha-1} J_{\alpha}(x)$$

$$= J_{\alpha}'(x) + J_{\alpha}'(x)$$

$$= 2 J_{\alpha}'(x)$$

Adding ③ and ④

$$J_{\alpha-1}(x) + J_{\alpha+1}(x) = (x^{\alpha} J_{\alpha})'(x) \cdot x^{-\alpha} - (x^{-\alpha} J_{\alpha})'(x) \cdot x^{\alpha}$$

$$\Rightarrow x^{-\alpha}((x)^{\alpha} J_{\alpha}'(x) + J_{\alpha}(x) \cdot \alpha \cdot x^{\alpha-1}) - x^{\alpha}(x^{-\alpha} J_{\alpha}'(x) - J_{\alpha}(x) \alpha (x^{-\alpha-1}))$$

$$= x^{-\alpha} x^{\alpha} J_{\alpha}'(x) + J_{\alpha}(x) \alpha \cdot x^{\alpha-1} \cdot x^{-\alpha} - x^{\alpha} \cdot x^{-\alpha} J_{\alpha}'(x) + x^{\alpha} \alpha x^{-\alpha-1} J_{\alpha}(x)$$

$$= J_{\alpha}(x) \alpha x^{-1} + \alpha x^{-1} J_{\alpha}(x)$$

$$= 2 J_{\alpha}(x) \alpha x^{-1}$$

$$J_{\alpha-1}(x) + J_{\alpha+1}(x) = 2\alpha x^{-1} J_{\alpha}(x)$$

Regular singular point

Definition

Induced equation

The equation $\tilde{L}(y) = t^4 y'' + [2t^3 - t^2 \tilde{a}_1(t)]y' + \tilde{a}_2(t)y = 0$ is called the induced equation associated with $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$

Definition :

We say that infinity is a regular singular point for $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ if the induced equation $\tilde{L}(y) = 0$ has the origin at $t=0$ as a regular singular point

1 Problem:- 1

V.I.P
(26)

a) Show that infinitely is a regular singular point for the legendre equation $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

b) compute the induced equation associated with legendre equation and the substitution $x = 1/t$

c) compute the indicial polynomial and its roots of the induced equation

Solution: a) Given $L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$$\Rightarrow y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

comparing the equation with

$$y'' + a_1(x)y' + a_2(x)y = 0$$

We get,

$$a_1(x) = \frac{-2x}{(1-x^2)} \; ; \; a_2(x) = \frac{\alpha(\alpha+1)}{1-x^2}$$

b) Now,

$$\tilde{a}_1(t) = a_1(1/t) = \frac{-2 \times 1/t}{1 - 1/t^2} = \frac{-2/t}{t^2 - 1/t^2}$$

$$= \frac{-2t}{t^2 - 1}$$

and $\tilde{a}_2(t) = a_2(1/t) = \dfrac{\alpha(\alpha+1)}{1 - 1/t^2} = \dfrac{t^2\alpha(\alpha+1)}{t^2 - 1}$

then the induced equation of $L(y) = 0$ is

$$t^4 y'' + [2t^3 - t^2 \tilde{a}_1(t)]y' + \tilde{a}_2(t) y = 0$$

$$t^4 y'' + \left\{2t^3 + \frac{t^2 2t}{t^2-1}\right\}y' + \frac{t^2\alpha(\alpha+1)}{t^2-1} y = 0 \quad \times (t^2-1)$$

$$t^4(t^2-1)y'' + \{2t^3(t^2-1) + 2t^3\}y' + t^2\alpha(\alpha+1)y = 0$$

$$t^4\{(t^2-1)y'' - (2t^5 - 2t^3 + 2t^3)y' + t^2\alpha(\alpha+1)y = 0$$

$$t^2\{t^2(t^2-1)y'' + 2t^3y' + \alpha(\alpha+1)y\} = 0$$

$$t^2(t^2-1)y'' + 2t^3y' + \alpha(\alpha+1)y\} = 0$$

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then the induced equation is

$$\div (t^2-1) \Rightarrow L(y) = t^2y'' + \frac{2t^3}{(t^2-1)}y + \frac{\alpha(\alpha+1)}{t^2-1}y = 0$$

we get,

$$\tilde{a}(t) = \frac{2t^3}{t^2-1} \text{ and } \tilde{b}(t) = \frac{\alpha(\alpha+1)}{t^2-1}$$

0 is a regular singular point for the induced equation $\tilde{L}(y) = 0$ ①

α is a regular singular point for the legendre equation $L(y) = 0$

b) $\tilde{L}(y) = t^2y'' + \frac{2t^3}{t^2-1}y' + \frac{\alpha(\alpha+1)}{t^2-1}y = 0$

C) The indical polynomial for $\tilde{L}(y) = 0$

$$q(r) = r(r-1) + \tilde{a}(0)r + \tilde{b}(0)$$

$$= r(r-1) + 0\cdot r + \frac{\alpha(\alpha+1)}{-1}$$

$$q(r) = r^2 - r - \alpha(\alpha+1)$$

$$q(r) = 0 \Rightarrow r^2 - r - \alpha(\alpha+1) = 0$$

$$r_1 = -\alpha \text{ and } r_2 = \alpha+1$$

The roots of indical polynomial are $-\alpha$ and $\alpha+1$

UNIT - IV

Existence and uniqueness of solutions to first order equation :

Equation with variable seperated

We consider the general first order equation
$y' = f(x, y)$ is some continuous function

The linear equation ①

$$y' - g(x)y + h(x)$$

Where g, h are continuous on some interval I, has a solution ϕ is of the form

$$\boxed{\phi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Qt} h(t)\,dt + c\,e^{Qx}}$$

Where $Q(x) = \int_{x_0}^{x} g(t)\,dt$

x_0 is the I and c is a constant

1) Solve $y' = y^2$ with initial condition $\phi(1) = -1$

solution: Given $y' = y^2$

$$\frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx$$

$$-\frac{1}{y} = x + c \qquad \frac{-1}{x+c} = y$$

$y = \phi(x)$ Given $\phi(1) = -1$

$$\phi(1) = \frac{-1}{1+c} \Rightarrow -1 = \frac{-1}{1+c}$$

$$1 + c = 1$$

$$\therefore c = 0 \quad \therefore y = -\frac{1}{x}$$

$$\text{ie) } \phi(x) = -1/x.$$

Definition :-

A first order equation $y' = f(x, y)$ is said to have the variables seperated of f can be written in the form $\boxed{f(x, y) = \frac{g(x)}{h(y)}}$

Where g, h are functions of a single argument

Theorem : 4.1

Let g, h be continuous real valued function for $a \le x \le b$, $c \le y \le d$ respectively. $y = \frac{g(x)}{h(y)}$

Proof

Consider the equation

$$h(y)y' = g(x) \rightarrow ①$$

If G, H are any functions such that $G' = g$, $H' = h$ and c is any constant such that the relation $H(y) = G(x) + c$ defines a real valued differentiable function ϕ for x in some Interval I contain in $a \le x \le b$ then ϕ will be solution of ① on I.

Conversely, If ϕ is a solution of ① in I, it satisfies the relation $H(y) = G(x) + c$ on I for constant c

① solve $y' = y^2$

solution

$$y' = y^2$$

Let $f(x,y) = y^2$

comparing with $f(x,y) = \frac{g(x)}{h(y)}$

$g(x) = 1$ and $h(y) = 1/y^2$

Clearly $h(y)$ is not continuous at $y = 0$

$$y' = y^2$$

$$\Rightarrow \frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + c$$

$$\Rightarrow y = \frac{-1}{x+c}$$

$\therefore$ The solution $\phi(x)$ is $\phi(x) = \frac{-1}{x+c}$ where c is con-
-ant

② Solve : $y' = 3y^{2/3}$ APr-18

Given $\frac{dy}{dx} = 3y^{2/3}$

$$\frac{dy}{y^{2/3}} = 3dx$$

$$\int dy \cdot y^{-2/3} = 3\int dx$$

$$\frac{y^{-2/3+1}}{-2/3+1} = 3x + c_1$$

$3y^{1/3} = 3x + c_1$

$y^{1/3} = x + c$

3

$y = (x+c)^3$

The solution $\phi(x)$ is $\phi(x) = (x+c)^3$ where c is constant

Definition: I.D Back side

A function f defined for real x, y said to be Exact in R if there exist a function F having continuous first partial derivatives such that

$$\frac{\partial F}{\partial x} = M \; ; \; \frac{\partial F}{\partial y} = N$$

Theorem: 1.2

Apr-19

Let M, N be two real valued Function which have continuous first partial derivatives on some rectangle $R: |x - x_0| \leq a \quad |y - y_0| \leq b$ then the equation $M(x,y) + N(x,y)y' = 0$ is exact in R iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Proof: Suppose $M(x,y) + N(x,y)y' = 0$ is exact and F is a function which has continuous second derivatives

$$\frac{\partial F}{\partial x} = M \; , \; \frac{\partial F}{\partial y} = N$$

then $$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y} \; ; \; \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

conversely, Suppose $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Let us consider the function F satisfying $\frac{\partial F}{\partial x} = M$

$\frac{\partial F}{\partial y} = N$

$$F(x,y) - F(x_0,y_0) = F(x,y) - F(x_0,y) + F(x_0,y) - F(x_0,y_0)$$

$$= \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y)\,ds + \int_{y_0}^{y} \frac{\partial F}{\partial y}(x_0,t)\,dt$$

$$= \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \rightarrow (1)$$

similarly

$$F(x,y) - F(x_0,y_0) = F(x,y) - F(x,y_0) + F(x,y_0) - F(x_0,y_0)$$

$$= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y_0)\,ds \quad ①$$

$$= \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds \rightarrow ②$$

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Now, Define F by

$$F(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \rightarrow ③ \quad [\text{from } ①]$$

$F(x_0,y_0) = 0$

and $\frac{\partial F}{\partial x}(x,y) = M(x,y)$ for all (x,y) in R

similarly

$$F(x,y) = \int_{y_0}^{y} N(x,t)\,dt \int_{x_0}^{x} M(s,y_0)\,ds \rightarrow ④$$

(From ②)

$\Rightarrow \frac{\partial F}{\partial y}(x,y) = N(x,y)$. For all (x,y) in R

We have to prove that equation ④ is valid where F is the Function given by ③

$$F(x,y) - \left[\int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds\right]$$

$$= \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt - \int_{y_0}^{y} N(x,t)\,dt - \int_{x_0}^{x} M(s,y_0)\,ds$$

$$= \int_{x_0}^{x} [M(s,y) - M(s,y_0)]\,ds - \int_{y_0}^{y} [N(x,t) - N(x_0,t)]\,dt$$

$$= \int_{x_0}^{x} \left[\int_{y_0}^{y} \frac{\partial M}{\partial y}(s,t)\,dt\right] ds - \int_{y_0}^{y} \left[\int_{x_0}^{x} \frac{\partial N}{\partial x}(s,t)\,ds\right] dt$$

$$= \int_{x_0}^{x} \int_{y_0}^{y} \left[\frac{\partial M}{\partial y}(s,t) - \frac{\partial N}{\partial x}(s,t)\right] dt \cdot ds$$

$$= 0 \qquad \left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right]$$

Hence Proved.

Home work Problem: unit - III

1. Find all real valued solution to the following equation

a) $y' = x^2 y$.

Soln: Given $y' = x^2 y$

$\frac{dy}{dx} = x^2 y$

Nov-18

$\frac{dy}{y} = x^2 dx$

$\int \frac{dy}{y} = \int x^2 dx$

$y = \phi(x)$

$\log y = x^3/3 + c_1$

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$y = e^{x^3/3 + c_1}$

$y = e^{x^3/3} \cdot e^{c_1}$

$\phi(x) = e^{x^3/3} \cdot c$

b) $yy' = x$

Soln: Given $yy' = x$

$y\, dy/dx = x$

$y\,dy = x\,dx$

$\int y\,dy = \int x\,dx$

$y^2/2 = x^2/2 + c$

$y^2 = x^2 + c$

y

$\frac{dy}{dx}$

$y = \sqrt{x^2 + c}$

$\phi(x) = \sqrt{x^2 + c}$

(c) $y' = \frac{x + x^2}{y + y^2}$ Apr-19

Given $y' = \frac{x + x^2}{y + y^2}$

Soln:

$\frac{dy}{dx} = \frac{x + x^2}{y + y^2} \Rightarrow dy\,(y + y^2) = (x + x^2)\,dx$

$\int (y + y^2)\,dy = \int (x + x^2)\,dx$

$y^2/2 + y^3/3 = x^2/2 + x^3/3 + c_1$

$3y^2 + 2y^3 = 3x^2 + 2x^3 + c$

) $y' = \frac{e^{x-y}}{1 + e^x}$

Solution: Given $\frac{dy}{dx} = \frac{e^x e^{-y}}{1 + e^x}$

$e^y dy = \frac{e^x}{1 + e^x} dx$

$$\int e^y dy = \int \frac{e^x}{1+e^x} dx$$

$$e^y = \log(1+e^x) + c$$

e) $y' = x^2y^2 - 4x^2$

Soln: Given $y' = x^2y^2 - 4x^2$

$y' = x^2(y^2-4)$ (6)

$$\frac{dy}{dx} = x^2(y^2-4)$$

$$\int \frac{dy}{(y^2-4)} = \int x^2 dx \quad \left[\because \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$\frac{1}{4}\log\left(\frac{y-2}{y+2}\right) = \frac{x^3}{3} + c$$

$$\log\left(\frac{y-2}{y+2}\right) = 4\frac{x^3}{3} + c$$

2) Show that the solution of $y' = y^2$ which passes through the point (x_0, y_0) is given by $\phi(x) = \frac{y_0}{1-y_0(x-x_0)}$

a) solution: $y' = y^2$

$$\frac{dy}{dx} = y^2$$

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$$\int \frac{dy}{y^2} = \int dx$$

$$\frac{-1}{y} = x + c$$

$$y = \frac{-1}{x+c} \rightarrow (1)$$

y^{-2}, $\frac{y^{-2+1}}{-2+1}$, $\frac{y^{-1}}{-1}$, $-y^{-1}$, $-\frac{1}{y}$.

at (x_0, y_0), $y_0 = \frac{-1}{x_0+c} \Rightarrow x_0 + c = \frac{-1}{y_0}$

$$c = \frac{-1}{y_0} - x_0 \Rightarrow c = -\left(x_0 + \frac{1}{y_0}\right)$$

$\therefore$ (1) becomes $y = \frac{-1}{x - \left(x_0 + \frac{1}{y_0}\right)}$

$$y = \frac{-y_0}{y_0(x-x_0)-1}$$

$$y = \frac{y_0}{1 - y_0(x-x_0)}$$

$\frac{y_0x - x_0y_0 - 1}{y_0}$ $= \frac{-1y_0}{y_0(x-x_0)-1}$

$\therefore$ The solution is $\phi(x) = \frac{y_0}{1-y_0(x-x_0)}$

b) For which x. Is ϕ as well defined function?

Soln:- ϕ is a well defined function if $y_0 = 0$ and all real x

(or)

if $y_0 \neq 0$ all real $x \neq x_0 + \frac{1}{y_0}$

c) For which x, Is ϕ a solution of the problem $y' = y^2$ $y(x_0) = y_0$?

Solution: ϕ is a solution of the given problem if $y_0 = 0$ and for all x

(or)

if $y_0 > 0$ $\quad -\infty < x < x_0 + 1/y_0$

(or)

if $y_0 < 0$ $\quad x_0 + \frac{1}{y_0} < x < \infty$

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Problem: 1

Solve: $y' = \frac{x+y}{x-y}$

Solution:

Let $y = vx$ $\qquad \frac{y}{x} = v$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x+vx}{x-vx} \qquad \frac{\cancel{x}(1+v)}{\cancel{x}(1-v)}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$x\frac{dv}{dx} = \frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$dv \cdot \frac{1-v}{1+v^2} = \frac{dx}{x}$$

$$\int\frac{dv}{1+v^2} - \int\frac{v\,dv}{1+v^2} = \int\frac{dx}{x}$$

$$\tan^{-1}v - \tfrac{1}{2}\log(1+v^2) = \log x + \log c_1$$

$$2\tan^{-1}v = \log(1+v^2) + 2\log x + 2\log c_1$$

$$2\tan^{-1}v = \log(1+y^2/x^2) + \log x^2 + \log c_1^2$$

$$2\tan^{-1}(y/x) = \log\left(\frac{x^2+y^2}{x^2}\right) + \log x^2 + \log c_1^2$$

$$= \log x^2\left(\frac{x^2+y^2}{x^2}\right)c_1^2\Big)$$

$$2\tan^{-1}(y/x) = \log c(x^2+y^2)$$

Definition:

The equation $y' = f(x,y)$ is homogenuous if f is homogenous of degree zero

i.e) $f(tx, ty) = f(x,y)$

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Remark:- 1

The equation $y' = f(x,y)$ is homogenuous then it can be reduced to ones with variables seperated

$$\text{Let } y = vx$$

$$y' = v + xv'$$

$$\therefore y \Rightarrow f(x,y) \Rightarrow v + xv' = f(x, vx)$$

$$\Rightarrow v + xv' = f(1, v)$$

$\therefore$ ($y' = 6(x,y)$ is homogenous

$$\Rightarrow xv' = f(1,v) - v$$

$$\Rightarrow v' = \frac{f(1,v) - v}{x}$$

which is an equation for v with variables seperated

Remark :- 2

Let us consider the first order equation of the form

$$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2} \rightarrow (1)$$ where a_1, a_2 b_1 b_2 c_1 and c_2 are constants. Then it can be reduced to the homogenous equation

Let $x = \xi + h$ and $y = \eta + k$

where h and k are constants

$$\therefore \frac{dx}{d\xi} = 1 \text{ and } \frac{dy}{d\eta} = 1$$

$$\Rightarrow \frac{dx}{d\xi} = \frac{dy}{d\eta} \Rightarrow \frac{dy}{dx} = \frac{d\eta}{d\xi}$$

$\therefore$ (1) becomes $$\frac{d\eta}{d\xi} = \frac{a_1(\xi + h) + b_1(\eta + k) + c_1}{a_2(\xi + h) + b_2(\eta + k) + c_2}$$

$$= \frac{a_1 \xi + b_1 \eta + b_1 k + a_1 \eta + c_1}{a_2 \xi + b_2 \eta + b_2 k + a_2 \eta + b_2 k + c_2}$$

If h and k satisfy the equation

$$a_1 h + b_1 k + c_1 = 0$$

$$a_2 h + b_2 k + c_2 = 0$$

then the equation (1) becomes homogenous otherwise either $u = a_1 x + b_1 y$ or $u = a_2 x + b_2 y$ leads to a seperation of variables

Eg: solve $y' = \dfrac{x-y+2}{x+y-1}$

Solution: Given $y' = \dfrac{x-y+2}{x+y-1}$

To find the value of h and k

$h-k+2=0 \rightarrow (1)$

$h+k-1=0 \rightarrow (2) \Rightarrow h = -1/2$

$2h+1$

$$\begin{array}{r} h-k+2=0 \\ h+k-1=0 \\ \hline 2h+1=0 \end{array}$$

put $\boxed{h=-1/2}$ in (2) we get

$-1/2 + k - 1 = 0 \Rightarrow \boxed{k = +3/2}$

(9)

$(1) \Rightarrow y' = \dfrac{x-y+(3/2+1/2)}{x+y-(3/2+1/2)}$

$$y' = \frac{(x+1/2)-(y-3/2)}{(x+1/2)+(y-3/2)} \rightarrow (2)$$

Let $X = x+1/2$, $Y = y-3/2$

$\Rightarrow \dfrac{dX}{dx} = 1 \quad \dfrac{dY}{dy} = 1$

$\therefore \dfrac{dX}{dx} = \dfrac{dY}{dx}$

$\therefore \dfrac{dY}{dX} = \dfrac{dy}{dx}$

(2) becomes

$$\frac{dY}{dX} = \frac{X-Y}{X+Y} \rightarrow (3)$$

Let $Y = VX \Rightarrow V = \dfrac{Y}{X}$

$\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$

$\therefore (3) \Rightarrow$

$$V + X\frac{dV}{dX} = \frac{X-VX}{X+VX}$$

$$V + X\frac{dV}{dX} = \frac{1-V}{1+V}$$

$$X\frac{dV}{dX} = \frac{1-V}{1+V} - V$$

$$X\frac{dV}{dX} = \frac{1-V-V-V^2}{1+V}$$

$$X\frac{dV}{dX} = \frac{1-2V-V^2}{1+V}$$

$$\frac{-2(1+v)}{2(1-2v-v^2)}\,dv = \frac{dx}{x}$$

Integrate

$$-\tfrac{1}{2}\log(1-2v-v^2) = \log x + \log c_1$$

$$\log(1-2v-v^2) + \log x^2 = \log c$$

$$x^2(1-2v-v^2)=c$$

$$x^2\left(1-\frac{2y}{x}-\frac{y^2}{x^2}\right)=c \qquad (10)$$

$$(x^2-2yx-y^2)=c$$

$$(x+\tfrac{1}{2})^2 - 2(x+\tfrac{1}{2})(y-\tfrac{3}{2}) - (y-\tfrac{3}{2})^2=c$$

Example:

Determine the following equation is exact and solve

$$y' = \frac{3x^2-2xy}{x^2-2y}$$

Solution Given that

$$y' = \frac{3x^2-2xy}{x^2-2y}$$

$$\Rightarrow (x^2-2y)y' = 3x^2-2xy$$

$$\Rightarrow (3x^2-2xy) - (x^2-2y)y'=0$$

Comparing with the equation

$$M(x,y)+N(x,y)y'=0$$

We get $M(x,y) = 3x^2-2xy$

$N(x,y) = -(x^2-2y)$

$$\frac{\partial M}{\partial y} = -2x \quad , \frac{\partial N}{\partial x} = -2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given equation is exact

Find the solution F satisfy the condition

$$\frac{\partial F}{\partial x} = M \; ; \; \frac{\partial F}{\partial y} = N$$

$$\therefore \frac{\partial F}{\partial x} = 3x^2-2xy$$

Integrate

$$F = 3\cdot \frac{x^3}{3} - 2y\frac{x^2}{2} + F(y)$$

$$F = x^3 - yx^2 + F(y) \rightarrow (1)$$

Diff. w.r to 'y'

$$\frac{\partial F}{\partial y} = -x^2 + F'(y)$$

$$N = -x^2 + F'(y) \quad \{\because \frac{\partial F}{\partial y} = N\}$$

$$-x^2 + 2y = -x^2 + F'(y)$$

$$f'(y) = 2y$$

Integrate $\Rightarrow$ $F(y) = y^2 + c$

① becomes

$$F(x,y) = x^3 - yx^2 + y^2 + c$$

Which is the required solution

(11)

$2xy\,dx + (x^2 + 3y^2)dy = 0$ Nov-18

Solution: Given that Ap-18

$$2xy\,dx + (x^2 + 3y^2)dy = 0$$

Comparing with the equation

$$M(x,y) + N(x,y)\,y' = 0$$

We get $M(x,y) = 2xy$

$$N(x,y) = x^2 + 3y^2$$

$$\frac{\partial M}{\partial y} = 2x \; ; \; \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the given equation is exactly

Find the solution F satisfying the conditions

$$\frac{\partial F}{\partial x} = M \; ; \; \frac{\partial F}{\partial y} = N$$

$$\frac{\partial F}{\partial x} = 2xy$$

Integrate

$$F = 2\frac{x^2}{2} \cdot y + F(y)$$

$$F = x^2 y + F(y)$$

Diff with r. to 'y'

$$\frac{\partial F}{\partial y} = x^2 + f'(y)$$

$$N = x^2 + f'(y)$$

$$x^2 + 3y^2 = x^2 + f'(y)$$

$$3y = f'(y)$$

Integrate $F(y) = 3y^3/3 + C$

$F(y) = y^3 + c$

① becomes

$F(x,y) = x^2y + y^3 + c$

Which is the required solution

② $(x^2 + xy)dx + xy\,dy = 0$

Solution. Given that

$(x^2 + xy)dx + xy\,dy = 0$

$M(x,y) = x^2 + xy$

$M(x,y) = xy$

$\frac{\partial M}{\partial y} = x \; ; \; \frac{\partial N}{\partial x} = y$

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$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence the given equation is not exact

③ $e^x dx + (e^y (y+1))dy = 0$

solution Given that

$e^x dx + (e^y(y+1))dy = 0$

$M(x,y) = e^x$

$N(x,y) = e^y(y+1)$

$\frac{\partial M}{\partial y} = 0 \; ; \; \frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given equation is exact

Find the solution F satisfying the conditions

$\frac{\partial F}{\partial y} = M \; ; \; \frac{\partial F}{\partial x} = N$

$\frac{\partial F}{\partial y} = e^x$

Integrate

$F = e^x + F(y) \rightarrow ①$

Diff wr 'to' y'

$\frac{\partial F}{\partial y} = 0 + F'(y)$

$N = F'(y)$

$e^y(y+1)dy = F'(y)$

Integrate

$\int e^y y\,dy + \int e^y dy = F(y)$

$y \cdot e^y - \int e^y dy + e^y = F(y)$

$y \cdot e^y - e^y + e^y = F(y)$

$y \cdot e^y = F(y)$

① becomes

$F = e^x + ye^y + c$

Which is the required solution (13)

Definition If the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact then exists a Function $u(x,y)$ Now, where zero, such that

$u(x,y)\,M(x,y)\,dx + u(x,y)\,N(x,y)\,dy = 0$ is exact

such a Function $u(x,y)$ is called Integrating factor

Example: Let us consider the equation $y\,dx - x\,dy = 0 \rightarrow ①$

For ① $M(x,y) = y$; $N(x,y) = -x$

$$\frac{\partial M}{\partial y} = 1 \; ; \; \frac{\partial N}{\partial x} = -1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

① is not exact

Let $u(x,y) = 1/y^2$

then $u(x,y)\,M(x,y) + u(x,y)\,N(x,y) = 0$

Becomes $\frac{1}{y^2}\,y\,dx - \frac{1}{y^2}\,x\,dx = 0 \rightarrow ②$

For ② $M(x,y) = 1/y$; $N(x,y) = -x/y^2$

$$\frac{\partial M}{\partial y} = -1/y^2 \; ; \; \frac{\partial N}{\partial x} = -1/y^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

② is exact

$$② \Rightarrow \frac{y\,dx - x\,dy}{y^2} = 0$$

$d(x/y) = 0$

$\Rightarrow x/y = c_1$

$\Rightarrow x = yc_1$

$\Rightarrow y = xc$

Which is required solution

Remark:

Consider $M(x,y)dx + N(x,y)dy = 0$ where M, N are continuous first partial derivative on some rectangle R.

Prove that a function u on R having continuous first partial derivatives is an integrating factor iff

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$$

Proof: The function u on R is an integrating factor if and only if

$u(x,y)\,M(x,y)dx + u(x,y)\cdot N(x,y)\,dy = 0$ is exact (By condition)

$$\frac{\partial}{\partial y}\{u(x,y)M(x,y)\} = \frac{\partial}{\partial x}\{u(x,y)N(x,y)\}$$

(by thm 4.2)

iff $u\frac{\partial M}{\partial y} + M\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$

iff $u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$ 14

Hence proved

Remark:- 2

a) If the equation $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor u which is a function of x alone then $p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a continuous function F, x alone

b) If p is a continuous and independent of y show $u(x) = e^{P(x)}$ where p is any function

$P' = p$

Remark:- 3

a) If the equation $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor u which is a function y alone. then $q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is continuous function y alone

If q is continuous and independent of x show that an integrating factor given by

$$u(y) = e^{Q(y)}$$

where Q is any function satisfying $\boxed{Q' = q}$

Determine the following equations are exact there and solve those (i) $x^2y^3dx - x^3y^2dy = 0$

Given that $x^2y^3dx - x^3y^2dy = 0$

HW

comparing with the equation

$M(x,y) + N(x,y)y' = 0$

$M(x,y) = x^2y^3$

$N(x,y) = x^3y^2$

$\frac{\partial M}{\partial y} = 3y^2x^2$

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$\frac{\partial N}{\partial x} = -3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Hence the given equation is not exact

2 $(x+y)dx + (x-y)dy = 0$

Solution Given $(x+y)dx + (x-y)dy = 0$

HW

$M(x,y) = x+y$, $N(x,y) = x-y$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the equation is exact

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = x+y$

$F = \frac{x^2}{2} + f(y) + xy$

$\frac{\partial F}{\partial y} = f'(y) + x$

$x - y = f'(y) + x$

$f'(y) = -y$

$f(y) = -y^2/2 + c$

$F = \frac{x^2}{2} + xy - y^2/2 + c$

$= x^2 + 2xy - y^2 + c$

3) HW $(2ye^{2x} + 2x\cos y)dx + (e^{2x} - x^2\sin y)dy = 0$

Solution Given that

$M(x,y)$ N $(2ye^{2x} + 2x\cos y)dx + (e^{2x} - x^2\sin y)dy = 0$

$M(x,y) = 2ye^{2x} + 2x\cos y$

$\frac{\partial M}{\partial y} = 2e^{2x} + (2x(-\sin y))$

$N(x,y) = e^{2x} - x^2 \sin y$

$\frac{\partial N}{\partial x} = 2e^{2x} - 2x \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the equation is exact

$\frac{\partial F}{\partial x} = M \quad , \quad \frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = 2ye^{2x} + 2x \cos y$

$F = \frac{2ye^{2x}}{2} + \frac{2x^2}{2} \cos y + f(y)$

$F = ye^{2x} + x^2 \cos y + f(y)$

$\frac{\partial F}{\partial y} = e^{2x} - x^2 \sin y + (f'(y))$

$e^{2x} - x^2 \sin y = e^{2x} - x^2 \sin y + f'(y)$

$f'(y) = 0$ 16

$f(y) = c$

$F = ye^{2x} + x^2 \cos y + c$

which is the required solution

Remark : 3 Example

1) Find an integrating factor of the following equation $(2y^3+2)dx + (3xy^2)dy = 0$ and solve them

Solution:

Given $(2y^3+2)dx + (3xy^2)dy = 0 \rightarrow (1)$

Hence $M(x,y) = 2y^3+2 \quad , N(x,y) = 3xy^2$

$\frac{\partial M}{\partial y} = 6y^2 \; ; \; \frac{\partial N}{\partial x} = 3y^2$

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then $q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{1}{2(y^3+1)} (3y - 6y^2)$

$= \frac{1}{2(y^3+1)} (-3y^2)$

$q = \frac{-3y^2}{2(y^3+1)}$ the function of y alone

Hence the integrating factor,

$u(y) = e^{Q(y)}$

where $Q(y) = \int q \, dy$

$Q(y) = \int \frac{-3y^2}{2(y^3+1)} dy$

$Q(y) = -\frac{1}{2} \log(1+y^3)$

$Q(y) = \log(1+y^3)^{-1/2}$

$u(y) = e^{Q(y)}$

$u(y) = (1+y^3)^{-1/2}$

$-\frac{1}{2}+1 : \frac{-1+2}{2} = \frac{1}{2}$

(1) becomes

$2(1+y^3)^{-1/2}(1+y^3)dx + (1+y^3)^{-1/2} 3xy^2 dy = 0$

$2(1+y^3)^{\frac{1}{2}} dx + (1+y^3)^{-1/2} 3xy^2 dy = 0 \rightarrow (2)$

For (2) $M = 2(1+y^3)^{1/2}$

$\frac{\partial M}{\partial y} = 2 \cdot \frac{1}{2} (1+y^3)^{\frac{1}{2}-1} \cdot 3y^2$

$\frac{\partial M}{\partial y} = (1+y^3)^{-1/2} \cdot 3y^2$

$N = (1+y^3)^{-1/2} 3xy^2$

$\frac{\partial N}{\partial x} = (1+y^3)^{-1/2} \cdot 3y^2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given equation is exact

Find the solution F satisfying the equation

$\frac{\partial F}{\partial x} = M \; ; \; \frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = 2(1+y^3)^{1/2}$

Integrate $F = 2(1+y^3)^{1/2} x + f(y) \rightarrow (3)$

Diff w.r to 'y'

$\frac{\partial F}{\partial y} = 2x \cdot \frac{1}{2} (1+y^3)^{-1/2} 3y^2 + f'(y)$

$(1+y^3)^{-1/2} 3xy^2 = (1+y^3)^{-1/2} 3xy^2 + f'(y)$

$f'(y) = 0 \Rightarrow f(y) = c$

$F = 2x(1+y^3)^{1/2} + c$

$2x(1+y^3)^{1/2} + c = 0$

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$\cos x \cos y \, dx - 2 \sin x \sin y \, dy = 0$

solution: Given,

HW

$\cos x \cos y \, dx - 2 \sin x \sin y \, dy = 0 \rightarrow (1)$

Hence $M(x,y) = \cos x \cos y \quad N = -2 \sin x \sin y$

$\frac{\partial M}{\partial y} = -\cos x \sin y \quad \frac{\partial N}{\partial x} = -2 \cos x \sin y$

$q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$= \frac{1}{\cos x \cos y} (\cos x \sin y - 2\cos x \sin y)$

$= \frac{-\cos x \sin y}{\cos x \cos y}$

$q = \frac{-\sin y}{\cos y} = \tan y$ the Function of y alone

Hence $u(y) = e^{Q(y)}$

where $Q(y) = \int q\, dy$

$= \int \frac{-\sin y}{\cos y} dy$ $\int \tan y\, dy$

$Q(y) = \log \cos y$

$Q(y) = e^{Q(y)}$ $e^{\log \cos y}$

$u(y) = \cos y$

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① becomes

$\cos x \cos y (\cos y) dx - 2 \sin x \sin y (\cos y) dy = 0$

$-\cos x \cos^2 y\, dx - 2 \sin x \sin y \cos y = 0 \rightarrow ②$

For ② $M = -\cos x \cos^2 y$ $N = -2 \sin x \sin y \cos y$

$\frac{\partial M}{\partial y} = -2 \cos x \cos y \sin y$ $\frac{\partial N}{\partial x} = -2 \sin y \cos x \cos y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given equation is exact

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = \cos x \cos^2 y$

Integrate $F = \cos^2 y \sin x + F(y) \rightarrow ③$

$\frac{\partial F}{\partial y} = 2\cos y (-\sin y) \sin y + F'(y)$

$-2 \sin x \cos y \sin y = -2 \cos y \sin y \sin x + f'(y)$

$F'(y) = 0$

$\therefore F(y) = c$

$F = \cos^2 y \sin x + c$

$\cos^2 y \sin x = c$

3) $(e^y + xe^y) dx + xe^y dy = 0$

Solution Given that

$(e^y + xe^y) dx + xe^y dy = 0$

Hence $M = e^y + xe^y \quad N = xe^y$

$\frac{\partial M}{\partial y} = e^y + xe^y \quad \frac{\partial N}{\partial x} = e^y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$P = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$

$= \frac{1}{xe^y}\left(e^y + xe^y - e^y\right) = 1$

$P = 1$ The function of x alone

$u(x) = e^{P(x)}$

$P(x) = \int P\,dx$

$= \int dx$

$P(x) = x$

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$u(x) = e^x$

① Becomes

$e^x (e^y + xe^y)dx + xe^x e^y dy = 0 \rightarrow ②$

$M = e^x (e^y + xe^y) \quad N = xe^x e^y$

$\frac{\partial M}{\partial y} = e^x \{e^y + xe^y\} \quad \frac{\partial N}{\partial x} = xe^y e^x + e^x e^y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given equation is exact

$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N$

$\frac{\partial F}{\partial x} = e^x e^y + x e^y e^y$

Integrate

$F = e^y e^x + e^y \{xe^x - e^x\} + f(y)$

$= e^y e^x + xe^x e^y - e^x e^y + f(y)$

$F = xe^x e^y + f(y)$

$\frac{\partial F}{\partial y} = xe^x e^y + f'(y)$

$xe^x e^y = xe^x e^y + f'(y)$

$f'(y) = 0$

$f(y) = c$

$F = xe^y e^x + c$

$c = xe^y e^x$

The Method of Successive Approximation

Definition

Let $y' = F(x,y)$ be a general first order equation where F is any continuous real valued function defined on some rectangle

$R: |x-x_0| \le a$; $|y-y_0| \le b$ $(a, b > 0)$ in the real (x,y) plane and there is a solution ϕ of $y' = F(x,y)$ satisfying $\phi(x_0) = y_0$ on some interval I containing x_0

Then there is a real valued differentiable function ϕ satisfying

$\phi(x_0) = y_0$ such that the point $(x, \phi(x))$ are in the R for all x in I and $\phi'(x) = F(x, \phi(x))$ for all x in I. such a function ϕ is called a solution to the initial value problem

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$$y' = F(x,y) ; \; y(x_0) = y_0 \text{ on } I$$

Theorem : 4.3

A function ϕ is a solution of the initial value Problem $y' = F(x,y), y(x_0) = y_0$ on an interval I iff it is a solution of the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t,y)\, dt \text{ on } I$$

Proof: Suppose ϕ is a solution of the initial value problem on I then

Given condition

$$\phi'(x) = F(x, \phi(x)) \text{ on } I \rightarrow ①$$

since ϕ is continuous on I and F is continuous on R

The Function F defined by $F(t) = F(t, \phi(t))$ is continuous on I

Integrate ① from x_0 to x We get

$$\int_{x_0}^{x} \phi'(t)\, dt = \int_{x_0}^{x} F(t, \phi(t))\, dt$$

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} F(t, \phi(t))dt$$

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} F(t, \phi(t))dt$$

since $\phi(x_0) = y_0$

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$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt$$

Here ϕ is a solution if $y = y_0 + \int_{x_0}^{x} f(t,y)dt$

conversely, suppose ϕ satisfies

$$y = y_0 + \int_{x_0}^{x} F(t,y)dt \qquad x = x_0 \quad y(x) = y_0 + \int_{x_0}^{x} f(t,y)dt$$

then $\phi(x) = y_0 + \int_{x_0}^{x} F(t, \phi(t))dt \qquad y(x_0) = y_0 + 0$

Differentiate

$$\phi'(x) = F(x, \phi(x)) \text{ for all } x \text{ on } I$$

Also $\phi(x_0) = y_0$

Hence ϕ satisfying the given initial value problem

Definition:

The Function $\phi_0, \phi_1, \phi_2, \ldots$ defined by

$\phi_0(x) = y_0$, $\boxed{\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))dt}$,

$\{k = 0, 1, 2, \ldots\}$

Where $\phi_k(x) \to \phi(x)$ as $k \to \infty$ is a successive approximation to a solution of the Integral equation $y = y_0 + \int_{x_0}^{x} F(t,y)dt$ (or)

The initial value problem

$$\boxed{y' = F(x,y), y(x_0) = y_0}$$

Problem:

Find the solution $\phi(x)$ of the first order equation $y' = xy$, $y(0) = 1$ by finding successive approximation solution.

Given $y' = xy$, $y(0) = 1$

The integral equation corresponding to this problem is

$$y = 1 + \int_0^x F(t, y)\,dt$$

The successive approximation are

$$\phi_0(x) = 1$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))\,dt$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F(t, \phi_0(t))\,dt$$

$y(0)$ $\phi_0(x) = 1$

$$= 1 + \int_0^x F(t, 1)\,dt$$

$$= 1 + \int_0^x t\,dt$$

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$$= 1 + \left[t^2/2\right]_0^x$$

$$\phi_1(x) = 1 + x^2/2$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} F(t, \phi_1(t))\,dt$$

$y' = xy$ apply

$$= 1 + \int_0^x F(t, 1 + t^2/2)\,dt$$

$$= 1 + \int_0^x (t + t^3/2)\,dt$$

$y' = (t)\left(\frac{2+t^2}{2}\right) = \frac{t(2+t^2)}{2} = \frac{2t}{2} + \frac{t^3}{2} = t + t^3/2$

$$= 1 + \left[t^2/2 + t^4/8\right]_0^x$$

$$= 1 + x^2/2 + x^4/8$$

$$\phi_2(x) = 1 + x^2/2 + 1/2\,(x^2/2)^2$$

In general,

$$\phi_k(x) = 1 + x^2/2 + 1/2!\,(x^2/2)^2 + \cdots + 1/k!\,(x^2/2)^k$$

as $k \to \infty$ $\phi_k(x) \to e^{x^2/2}$ for all real x

Hence $\phi(x) = e^{x^2/2}$ is the solution of the given initial problem

2) consider the initial value problem APr-18

$y' = 3y + 1 \quad , y(0) = 2$ NOV-18

i) compute the first four approximation $\phi_0, \phi_1, \phi_2, \phi_3$

(ii) compute the solution $\phi(x)$

solution:- Given $y' = 3y+1 \quad y(0)=2$

The integral equation corresponding to this problem

$$y = 1 + \int_0^x F(t,y)\,dt$$

The successive approximation are

$$\phi_0(x) = 2$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))\,dt$$

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$$\phi_1(x) = y_0 + \int_{x_0}^{x} F(t, \phi_0(t))\,dt$$

$$= y_0 + \int_{x_0}^{x} F(t,2)\,dt$$

$$= 2 + \int_0^x (3(2)+1)\,dt$$

$$= 2 + 7[t]_0^x$$

$$= 2 + 7x$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} (t, \phi_1(t))\,dt$$

$$= 2 + \int_0^x f(t, 2+7t)\,dt$$

$$= 2 + \int_0^x (3(2+7t)+1)\,dt$$

$$= 2 + \int_0^x (7+21t)\,dt = 2 + \left[7t + 21\tfrac{t^2}{2}\right]_0^x$$

$$= 2 + 7x + 21x^2/2$$

$$\phi_3(x) = y_0 + \int_0^x F\left(t, 2+7t+\frac{21t^2}{2}\right)dt$$

$$= 2 + \int_0^x \left(3\left(2+7t+\frac{21t^2}{2}\right)+1\right)dt$$

$$= 2 + \int_0^x (7 + 21t + 63t^2/2)\,dt$$

$$= 2 + \left[7x + 21\frac{x^2}{2} + \frac{63}{3}x^3\right]$$

$$\phi_3(x) = 2 + 7x + \frac{21}{2}x^2 + \frac{63}{3}x^3$$

$$\phi_4(x) = y_0 + \int_{x_0}^x F(t, \phi_3(t))\,dt$$

$$= y_0 + \int_0^x F\left(t, 2 + 7t + \frac{21}{2}t^2 + \frac{63t^3}{3}\right)dt$$

$$= 2 + \int_0^x \left(3\left(2 + 7t + 21t^2/2 + \frac{63t^3}{3}\right) + 1\right)dt$$

$$= 2 + 7 + 21t + 63t^2/2 + \frac{189t^3}{3}$$

$$= 2 + 7x + \frac{21x^2}{2} + 63x^3/3 + \frac{189x^4}{4}$$

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$$\frac{dy}{dx} = 3y + 1$$

$$\frac{dy}{3y+1} = dx$$

$$1/3 \log(3y+1) = x + c$$

$$\log(3y+1) = 3x + c \rightarrow ①$$

At (x_0, y_0)

$$\log(3y_0 + 1) = 3x_0 + c$$

$$\log 7 = c$$

① Becomes

$$\log(3y+1) = 3x + \log 7$$

$$\log\left(\frac{3y+1}{7}\right) = 3x$$

$$\frac{3y+1}{7} = e^{3x}$$

$$3y = 7e^{3x} - 1$$

$$y = \frac{7e^{3x} - 1}{3}$$

Remark:

If F is continuous on R and Bounded then there exists $M>0$ such that $|F(x,y)| \le M$ for all x in R

Theorem: 4.4

The successive approximation ϕ_k defined by $\phi_0(x) = y_0 \cdot \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt \quad (k=0,1,2,\ldots)$ exist as continuous functions on I $|x-x_0| \le \alpha = \min\{a, b/M\}$ and $(x, \phi_k(x))$ is in R for all x in I indeed the ϕ_k satisfy $|\phi_k(x) - y_0| \le M|x-x_0|$ for x in I

Proof:

If $k=0$, $\phi_0(x) = y_0$

Hence, clearly ϕ_0 exists on I as a continuous Function and satisfies $|\phi_k(x) - y_0| \le M|x-x_0|$.

Now, $\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\, dt$

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$$\Rightarrow |\phi_1(x) - y_0| = \left| \int_{x_0}^{x} f(t, y_0)\, dt \right|$$

$$\Rightarrow |\phi_1(x) - y_0| \le \int_{x_0}^{x} |f(t, y_0)|\, dt$$

$$\Rightarrow |\phi_1(x) - y_0| \le \int_{x_0}^{x} M\, dt = M|x-x_0|$$

$$\Rightarrow |\phi_1(x) - y_0| \le M|x-x_0|$$

which shows that ϕ_1 satisfies the condition

Since f is continuous on R,

the Function F_0 defined by $F_0(t) = f(t, y_0)$ continuous on I

Thus $\phi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)\, dt$

Assume that the theorem is true for the Function $\phi_0, \phi_1, \phi_2, \ldots \phi_k$

we know that $(t, \phi_k(t))$ is in R, for t in I.

Thus, the function F_k is given by

$F_k(t) = f(t, \phi_k(t))$ exists for t in I

since f is continuous on R

F_k is also continuous on I and ϕ_k is a continuous function on I

$\therefore \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)\,dt$ exists as a continuous function on I

moreover, $|\phi_{k+1}(x) - y_0| = \left|\int_{x_0}^{x} F_k(t)\,dt\right|$

$$\leq \int_{x_0}^{x} |F_k(t)|\,dt$$

$$\leq M \int_{x_0}^{x} dt$$

$|\phi_{k+1}(x) - y_0| \leq M|x - x_0|$, for all x in I

Hence proved

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Note:

since for x in I, $|x - x_0| \leq \min\{a, b/M\} = b/M$

then $|\phi_k(x) - y_0| \leq M|x - x_0|$

$\Rightarrow |\phi_k(x) - y_0| \leq M \cdot b/M = b$

$\therefore |\phi_k(x) - y_0| \leq b$, for all x in I

Which shows that the point $(x, \phi_k(x))$ in R for all x in I

Also, the graph of each ϕ_k lines in the region T in R bounded by the two lines $y - y_0 = M(x - x_0)$ and $(y - y_0) = -M(x - x_0)$ and $x - x_0 = \alpha$, $x - x_0 = -\alpha$

1) compute the first four approximation $\phi_0, \phi_1, \phi_2, \phi_3$

i) $y' = x^2+y^2$, $y(0)=0$

solution

Given $y' = x^2+y^2$, $y(0)=0$ $x_0=0, y_0=0$

The integral domain corresponding to this problem is $y = 1 + \int_0^x F(t,y)\,dt$

The successive approximation are

$$\phi_0(x) = 0$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))\,dt$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F(t, \phi_0(t))\,dt$$

$F(x,y) = y'$
$= x^2+y^2$

$$= 0 + \int_0^x F(t,0)\,dt$$

$$= \int_0^x t^2\,dt$$

$$\phi_1(x) = \left[t^3/3 \right]_0^x = x^3/3$$

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$$\phi_2(x) = y_0 + \int_{x_0}^{x} F(t, \phi_1(t))\,dt$$

$$= 0 + \int_0^x F(t, t^3/3)\,dt$$

$$= \int_0^x \left[t^2 + t^6/6\right] dt$$

$$= \left[t^3/3 + x^6/6 \right]_0^x$$

$$\phi_2(x) = x^3/3 + x^7/63 = x^3/3 + x^7/63$$

$$\phi_3(x) = y_0 + \int_{x_0}^{x} F(t, \phi_2(t))\,dt$$

$$= 0 + \int_0^x F(t, t^3/3 + t^7/6)\,dt$$

$$= \int_0^x \left(t^2 + t^6/9 + t^4/3969 + 2t^{10}/189 \right) dt$$

$= x^3/3 + x^7/63 + x^{15}/59535 + 2x^{11}/2079$

$\phi_3(x) = x^3/3 + x^7/63 + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}$

(ii) $y' = 1 + xy$, $y(0) = 1$ Apr-19

Solution: Given that $y' = 1 + xy$, $y(0) = 1$

The integral equation corresponding to this Problem is

$$\phi_0(x) = 1$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))dt$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F(t, \phi_0(t))dt$$

$$= 1 + \int_0^x F(t, 1)dt$$

$$= 1 + \int_0^x (1+t)dt$$

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$$\phi_1(x) = 1 + x + x^2/2$$

$$\phi_2(x) = y_0 + \int_{x_0}^{x} F(t, \phi_1(t))dt$$

$$= 1 + \int_{x_0}^{x} F(t, 1 + t + t^2/2) dt$$

$$= 1 + \int_0^x 1 + t \times (1 + t + t^2/2) dt$$

$$= 1 + \int_0^x (1 + t + t^2 + t^3/2) dt$$

$$= 1 + \left[t + t^2/2 + t^3/3 + t^4/8 \right]_0^x$$

$$\phi_2(x) = 1 + x + x^2/2 + x^3/3 + x^4/8$$

$$\phi_3(x) = y_0 + \int_{x_0}^{x} F(t, \phi_2(t))dt$$

$$\phi_3(x) = y_0 + \int_{x_0}^{x} F(t, \phi_2(t))$$

$$= 1 + \int_0^x F(t, 1 + t + t^2/2 + t^3/3 + t^4/8) dt$$

$$= 1 + \int_0^x (1 + t + t^2 + t^3/2 + t^4/3 + t^5/8)\,dt$$

$$= 1 + \left[t + t^2/2 + t^3/3 + t^4/8 + t^5/15 + t^6/48\right]_0^x$$

$$\phi_3(x) = 1 + x + x^2/2 + x^3/3 + x^4/8 + x^5/15 + x^6/48$$

iii) $y' = y^2 \quad y(0) = 1 \quad y(x_0) = y_0$

Solution Given that $y' = y^2$, $y(0) = 1$

The Integral equation Corresponding to this Problem is $y = 1 + \int_0^x F(t, y)\,dt$

The sucessive approximation are

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$$\phi_0(x) = 0$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F(t, \phi_k(t))\,dt$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F(t, \phi_0(t))\,dt$$

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$$= 0 + \int_0^x F(t, 0)\,dt$$

$$= \int_0^x 0$$

$$\phi_1(x) = 0 \quad , \cancel{\phi_2(x) = 0, \phi_3(x) = 0}$$

The Lipschitz condition:-

Definition

Let f be a function defined for (x, y) in a set s. We say that f satisfies a Lipschitz condition on s if there exists a constants $k > 0$ such that

$$| f(x, y_1) - f(x, y_2) | \leq k\, | y_1 - y_2 |$$

For all $(x, y_1)(x, y_2)$ in s the constant k is called Lipschitz constant

Theorem: 4.5

Suppose S is either a rectangle $|x-x_0|\le a$, $|y-y_0|\le b$ $(a,b>0)$ or a strip $|x-x_0|\le a$, $|y|<\infty$ $(a>0)$ and that f is real valued function defined on $\frac{\partial f}{\partial y}$ exists is continuous valued f on S $|\frac{\partial f}{\partial y}(x,y)|\le k$ (x,y) in S $\to$ ①

For some $k>0$ then f satisfies a lipschitz condition on S with lipschitz constant k.

Proof:

We have $\int_{y_2}^{y_1}\frac{\partial f}{\partial y}(x,t)\,dt$

$$= f(x,y_1) - f(x,y_2)$$

$$\Rightarrow |f(x,y_1)-f(x,y_2)| = \left|\int_{y_2}^{y_1}\frac{\partial f}{\partial y}(x,t)\,dt\right|$$

$$\le \int_{y_2}^{y_1}\left|\frac{\partial f}{\partial y}(x,t)\,dt\right|$$

$$\le k\int_{y_2}^{y_1}dt \quad = k|y_1-y_2|$$

$\therefore |f(x,y_1)-f(x,y_2)|\le k\,|y_1-y_2|$ for all $(x,y_1)(x,y_2)$ in S

Hence f satisfies a lipschitz condition

Hence proved

Example:

Prove that $f(x,y)=xy^2$ satisfies a lipschitz condition

Solution:

Let $f(x,y)=xy^2$

$$\frac{\partial f}{\partial y}(x,y) = 2xy$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = |2xy| = 2\,|x|\,|y|$$

i) If R: $|x|\le 1$; $|y|\le 1$

Hence $\left|\frac{\partial f}{\partial y}(x,y)\right|\le 2$ for (x,y) on R

Hence the given function satisfies the lipschitz condition with Lipschitz constant 2.

ii) If $S: |x| \le 1, |y| < \infty$

Now,
$$\left| \frac{f(x,y) - f(x,0)}{y_1 - 0} \right| = \left| \frac{x y_1^2 - 0}{y_1} \right|$$
$$= |x y_1|$$
$$= |x||y_1|$$

If $|x| \ne 0$ as $|y_1| \to \infty$

$\left| \frac{\partial f}{\partial y}(x,y) \right|$ tends to ∞

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Hence the given function does not satisfy a lipschtiz condition on the strip $S: |x| \le 1, |y| < \infty$

2) prove that there exists a continuous function which is not satisfying a lipschitz condition on a rectangle.

Proof: Let $f(x,y) = y^{2/3}$ on $R: |x| \le 1; |y| \le 1$

If $y_1 > 0$
$$\left| \frac{f(x, y_1) - f(x,0)}{y_1 - 0} \right| = \left| \frac{y_1^{2/3} - 0}{y_1} \right|$$
$$= |y_1^{-1/3}|$$
$$= 1/y_1^{1/3}$$

As $y_1 \to 0$ $\left| \frac{\partial f}{\partial y}(x,y) \right|$ tends to ∞

Hence the function $f(x,y) = y^{2/3}$ does not satisfy a lipschitz condition on R.

Convergence of the successive approximation :-

Theorem : 4.6

Existence theorem for successive approximation

Let f be a continuous real valued function on the rectangle $R: |x - x_0| \le a, |y - y_0| \le b$ $(a, b) > 0)$

and let $|f(x,y)| \le M$ for all (x,y) in R. Further suppose that f satisfies a lipschitz condition with constant k in R. then the successive approximation, $\phi_0(x) = y_0$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt \quad (k = 0,1,2,\ldots)$$

converge on the interval $I: |x - x_0| \le \alpha = \min\{a, b/M\}$ to a solution ϕ of the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \text{ in } I$$

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Proof:- Step:-1

Convergence of $\{\phi_k(x)\}$

Now, $\phi_k = \phi_k + (\phi_0 - \phi_0) + (\phi_1 - \phi_1) + \cdots + (\phi_{k-1} - \phi_{k-1})$

$$= \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots + \phi_k - \phi_{k-1}$$

$$\therefore \phi_k(x) = \phi_0(x) + \sum_{p=1}^{k} (\phi_p - \phi_{p-1})$$

Hence $\phi_k(x)$ is a partial sum for all series

$$= \phi_0(x) + \sum_{p=1}^{\infty} (\phi_p - \phi_{p-1}) \rightarrow (*)$$

Theorem 4.3 show that,

The function ϕ_p exists as continuous function on I for all p and $(x, \phi_p(x))$ is in R for all x in I.

Moreover

$$|\phi_1(x) - \phi_0(x)| \le M|x - x_0| \rightarrow (A) \text{ for all } x \text{ in } I$$

We know that

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\,dt \rightarrow (1)$$

and

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt \rightarrow (2)$$

(1) - (2)

① - ② $k=0$

$$\phi_2(x) - \phi_1(x) = \int_{x_0}^{x} (f(t, \phi_1(t)) - f(t, \phi_0(t))] dt$$

$$\Rightarrow |\phi_2(x) - \phi_1(x)| = \int_{x_0}^{x} f(t, \phi_1(t)) - f(t, \phi_0(t)) dt$$

$$\leq \int_{x_0}^{x} |f(t, \underset{y_1}{\phi_1(t)}) - f(t, \underset{y_2}{\phi_0(t)})| dt \rightarrow ③$$

Since f satisfies the lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq k|y_1 - y_2|$$

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∴ ③ Becomes

$$|\phi_2(x) - \phi_1(x)| \leq \int_{x_0}^{x} k\, |\phi_0(t) - \phi_1(t)| dt$$

$$\leq k \int_{x_0}^{x} M\, |t - x_0| dt \quad \text{By } Ⓐ$$

$$= kM \int_{x_0}^{x} |t - x_0| dt \quad \text{if } x_0 \leq x$$

$$= kM \left[\frac{|t - x_0|^2}{2} \right]_{x_0}^{x}$$

$$= kM \frac{|x - x_0|^2}{2}$$

$$\therefore |\phi_2(x) - \phi_1(x)| \leq kM \frac{|x - x_0|^2}{2} \rightarrow Ⓑ$$

The result is also valid in case $x \leq x_0$

We have to prove that

$$|\phi_p(x) - \phi_{p-1}(x)| \leq \frac{M k^{p-1} |x - x_0|^p}{p!} \text{ for all } x \text{ in } I \rightarrow Ⓒ$$

By Induction,

The resut is true for $p=1$ & $p=2$ by Ⓐ & Ⓑ

Assume that the result Ⓒ is true for $p=m$

Let $p = m+1$

By the definition, successive approximation

$$\phi_{(m+1)}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_m(t)) dt \rightarrow ④$$

and $\phi_m(x) = y_0 + \int_{x_0}^{x} f(t, \phi_{m-1}(t))\,dt \rightarrow (5)$

(4) - (5) $\Rightarrow \left|\phi_{m+1}(x) - \phi_m(x)\right| = \left| \int_{x_0}^{x} [f(t, \phi_m(t)) - f(t, \phi_{m-1}(t))]\,dt \right|$

$\leq \int_{x_0}^{x} \left| f(t, \phi_m(t)) - f(t, \phi_{m-1}(t)) \right| dt$

(using Lipschtiz condition)

$\leq \int_{x_0}^{x} k \left| \phi_m(t) - \phi_{m-1}(t) \right| dt$

$\leq k \int_{x_0}^{x} \frac{M k^{m-1} |t - x_0|^m}{m!}$ (By induction hypothesis)

$\leq \frac{M k^m}{m!} \int_{x_0}^{x} |t - x_0|^m\,dt$

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$= \frac{M k^m}{m!} \left[\frac{|t - x_0|^{m+1}}{m+1} \right]_{x_0}^{x}$

$= \frac{M k^m}{(m+1)!} |x - x_0|^{m+1}$

$\therefore \left| \phi_{m+1}(x) - \phi_m(x) \right| \leq \frac{M k^m |x - x_0|^{m+1}}{(m+1)!}$

Hence the result is true for all $p = 1, 2, \ldots$

Hence (4) Becomes

$|\phi_k(x)| = \left| \phi_0(x) + \sum_{p=1}^{\infty} (\phi_p(x) - \phi_{p-1}(x)) \right|$

$\leq |\phi_0(x)| + \sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)| \rightarrow (6)$

clearly (6) shows that the infinite series $\phi_0(x) + \sum_{p=1}^{\infty} (\phi_p(x) - \phi_{p-1}(x))$ is absolutely converges on I

ie) the series $|\phi_0(x)| + \sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)|$

is converges on I

From ⓒ,

$$|\phi_p(x)-\phi_{p-1}(x)| \le \sum_{p=1}^{\infty} \frac{Mk^{p-1}|x-x_0|^p}{p!}$$

$k^{p-1} = k^p . k^{-1}$

$k^p (x-$

M/k

$$\le \frac{M}{k} \sum_{p=1}^{\infty} \frac{k^p |x-x_0|^p}{p!}$$

$e^x = 1 + x + x^2/2 + \cdots$

$= \sum_{k=0}^{\infty} x^k/k!$

$$|\phi_p(x)-\phi_{p-1}(x)| = \frac{M}{k} e^{k|x-x_0|}$$

Hence $\{\phi_k(x)\}$ is convergent on I

$\therefore \phi_k(x) \to \phi(x)$ as $k \to \infty$ for each x in I

Step : 2

Properties of the limit ϕ 35

First we prove that ϕ is continuous on I

If x_1, x_2 are in I

Successive app. defn

$$\phi_{k+1}(x_1) = y_0 + \int_{x_0}^{x_1} f(t, \phi_k(t))dt$$

$$\phi_{k+1}(x_2) = y_0 + \int_{x_0}^{x_2} f(t, \phi_k(t))dt$$

$$\phi_{k+1}(x_1) - \phi_{k+1}(x_2) = \int_{x_0}^{x_1} f(t, \phi_k(t))dt - \int_{x_0}^{x_2} f(t, \phi_k(t))dt$$

$$|\phi_{k+1}(x_1) - \phi_{k+1}(x_2)| \le \int_{x_2}^{x_1} |f(t, \phi_k(t))|dt$$

Continuous

$\to |x_1 - x_2| < \delta$

$\to |f(x_1) - f(x_2)| < \epsilon$

$$\le M \int_{x_2}^{x_1} dt = M|x_1 - x_2|$$

As $k \to \infty$

$$|\phi(x_1) - \phi(x_2)| \le M|x_1 - x_2|$$

ie) $\phi(x_1) \to \phi(x_2)$ as $x_1 \to x_2$

Let $x_1 = x$ and $x_2 = x_0$ then

$\phi(x) \to \phi(x_0)$ as $x \to x_0$

y_0

ie) $|\phi(x) - y_0| \le M|x - x_0|$, x in I

The point $(x, \phi(x))$ are in R for x in I

Step :- 3

Estimate for $|\phi(x) - \phi_k(x)|$

Now, $\phi(x) = \phi_0(x) + \sum_{P=1}^{\infty} [\phi_P(x) - \phi_{P-1}(x)]$ and

$$\phi_K(x) = \phi_0(x) + \sum_{P=1}^{k} [\phi_P(x) - \phi_{P-1}(x)]$$

$$\therefore \phi(x) - \phi_k(x) = \sum_{P=k+1}^{\infty} [\phi_P(x) - \phi_{P-1}(x)]$$

$$|\phi(x) - \phi_k(x)| \le \sum_{P=k+1}^{\infty} |\phi_P(x) - \phi_{P-1}(x)|$$

$|x - x_0| \le \alpha$ (min)

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$$\le \sum_{P=k+1}^{\infty} \frac{M \cdot k^P |x - x_0|^P}{kP!}$$

$$\le \frac{M}{k} \sum_{P=k+1}^{\infty} \frac{k^P \alpha^P}{P!}$$

$$= \frac{M}{k} \left\{ \frac{(k\alpha)^{K+1}}{(K+1)!} + \frac{(k\alpha)^{K+2}}{(K+2)!} + \frac{(k\alpha)^{K+3}}{(K+3)!} + \cdots \right\}$$

$$= \frac{M}{k} \frac{(k\alpha)^{K+1}}{(K+1)!} \left\{ 1 + \frac{k\alpha}{k+2} + \frac{(k\alpha)^2}{(K+2)(K+3)} + \cdots \right\}$$

$$< \frac{M}{k} \frac{(k\alpha)^{K+1}}{(K+1)!} \left\{ 1 + \frac{k\alpha}{1} + \frac{(k\alpha)^2}{2!} + \cdots \right\}$$

$k+2 > 1$, $\frac{1}{k+2} < 1$; $k+3 > 2$, $\frac{1}{k+3} < 2$

$$\therefore |\phi(x) - \phi_K(x)| < \frac{M}{k} \frac{(k\alpha)^{K+1}}{(K+1)!} e^{k\alpha}$$

Let $\varepsilon_k = \frac{(k\alpha)^{K+1}}{(K+1)!}$

As $k \to \infty$, $\varepsilon_k \to 0$

$$|\phi(x) - \phi_k(x)| \le \frac{M}{k} e^{k\alpha} \varepsilon_k,$$

where $\varepsilon_k \to 0$ as $k \to \infty$ $\quad y = y_0 + \int_{x_0}^{x} f(t,$

Step :- 4

The limit ϕ is a solution

Proof

We have to prove that $\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))$

for all x in I $\to$ (7)

since $f(t, \phi(t))$ is continuous

The right hand side of (7) is continuous on R and the function

$f(t) = f(t, \phi(t))$ is also continuous on

(5)

Now, $\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt$ and

same

$\phi_{k+1}(x) \to \phi(x)$ as $k \to \infty$ (37)

From (7) it is enough to prove that

$$\int_{x_0}^{x} f(t, \phi_k(t))\,dt \to \int_{x_0}^{x} f(t, \phi(t))\,dt \text{ as } t \to \infty$$

We have,

$$\left|\int_{x_0}^{x} f(t, \phi(t))\,dt - \int_{x_0}^{x} f(t, \phi_k(t))\,dt\right|$$

$$= \left|\int_{x_0}^{x} f(t, \phi(t)) - f(t, \phi_k(t))\,dt\right|$$

$$\le \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \phi_k(t))|\,dt$$

$$\leq k \int_{x_0}^{x} |\phi(t) - \phi_k(t)| \, dt$$ (By Lipschitz condition)

$$\leq k \frac{M}{k} e^{k\alpha} \varepsilon_k \, dt$$ step. 3 Ans

$$\therefore \left| \int_{x_0}^{x} f(t, \phi(t))\, dt - \int_{x_0}^{x} f(t, \phi_k(t))\, dt \right|$$

$$\leq M \cdot e^{k\alpha} \varepsilon_k \, |x - x_0| \quad (3.8)$$

(v) As $k \to \alpha$

$$M \cdot e^{k\alpha} \varepsilon_k \, |x - x_0| \to 0 \quad (\text{since } \varepsilon_k \to 0)$$

Hence proved

Theorem : 4.17   10m Nov-18

The k^{th} successive approximation ϕ_k to the solution ϕ of the initial value problem

$y' = f(x, y)$, $y(x_0) = 0$ satisfies

$$|\phi(x) - \phi_k(x)| \leq \frac{M}{k} \frac{(k\alpha)^{k+1}}{(k+1)!} e^{k\alpha} \text{ for all } x \text{ in } I$$

Ans : step: 1, 2, 3

UNIT-5 (1)

Non local existance of solutions:

Thrm 5.1

Let f be a real valued continuous function on the strip.

$S: |x - x_0| \leq a$, $|y| < \infty$, $(a > 0)$ and suppose that f satisfies on S a lipschitz condition which constant $k > 0$. The successive approximation $\{\phi_n\}$ for the pbrm $y' = f(x, y)$, $y(x_0) = y_0$ exists on the entire interval $|x - x_0| \leq a$ and converge these to a solution ϕ of (1).

proof:-

The successive approximations are given by

$$\phi_0(x) = y_0$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt \quad , k = 0, 1, 2 \ldots$$

(2) by thrm 4.4.

for $|x-x_0| \leq a$, each ϕ_K can be exists.

Since f is Continuous on S.

Then the function f_0 is given by

$f_0(x) = f(x, y_0)$ is continuous for $|x-x_0| \leq a$ and also bounded for $|x-x_0| \leq a$

let M be any positive costant.

Such that

$$|f(x, y_0)| \leq M \quad (|x-x_0| \leq a) \rightarrow (2)$$

By thrm 4.5

$\{\phi_K(x)\}$ converges and

$$\phi_1(x) = \phi_0(x) + \int_{x_0}^{x} f(t, y_0)\,dt$$

$$\Rightarrow |\phi_1(x) - \phi_0(x)| \leq \int_{x_0}^{x} |f(t, y_0)|\,dt$$

$$\Rightarrow |\phi_1(x) - \phi_0(x)| \leq M|x-x_0| \quad [\text{by } (2)]$$

Also, $|\phi_p(x) - \phi_{p-1}(x)| \leq \dfrac{MK^{p-1}}{p!}|x-x_0|^p \rightarrow (3)$

$$\Rightarrow |\phi_K(x) - y_0| \leq \left| \sum_{p=1}^{K} (\phi_p(x) - \phi_{p-1}(x)) \right|$$

$$\leq \sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)|$$

$$\leq \sum_{p=1}^{\infty} \frac{M}{K} \frac{K^p |x-x_0|^p}{p!}$$

$$\leq \frac{M}{K} \sum_{p=1}^{\infty} \frac{(Ka)^p}{p!}$$

$$\leq \frac{M}{K}(e^{Ka}-1) \text{ for } |x-x_0| \leq a$$

Let $b = \dfrac{M}{K}(e^{Ka}-1)$

(3) $\therefore |\phi_k(x) - y_0| \le b \quad (|x - x_0| \le a)$

As $k \to \infty$, we get

$$|\phi(x) - y_0| \le b, \quad (|x - x_0| \le a)$$

Since f is continuous on,

$$R : |x - x_0| \le a, \quad |y - y_0| \le b$$

and it is bounded.

ie) there is a positive constant N

Such that, $|f(x,y)| \le N$ for (x,y) in R

By Step 2: of the thrm 4.5

for x_1, x_2 in $|x - x_0| \le a$

$$|\phi_{k+1}(x_1) - \phi_{k+1}(x_2)| = \left| \int_{x_2}^{x_1} f(t, \phi_k(t))\, dt \right|$$

$$\le N\, |x_1 - x_2|$$

As $k \to \infty$,

$$|\phi(x_1) - \phi(x_2)| \le N |x_1 - x_2|$$

Hence proved by the thrm 4.5

Corollary: Suppose f is a real valued continuous function on the plane. $|x| < \infty$, $|y| < \infty$ which Satisfies a lipschitz condition on each strip

$$S_a : |x| \le a, \quad |y| \le \infty$$

where a is a positive number. then every initial value problem.

$$y' = f(x,y), \quad y(x_0) = y_0$$ has a soln which exists for all real x.

(4) proof: If x is any real number there is an $a > 0$ such that x contained inside an interval $|x - x_0| \leq a$

by thm 5.1

for this a function f satisfies the conditions on the strip $|x - x_0| \leq a$, $|y| \leq \infty$

Now, $$|x| - |x_0| \leq |x - x_0| \leq a$$

$$|x| \leq |x_0| + a$$

Since this strip is contained in the strip.

$$|x| \leq |x_0| + a, \quad |y| < \infty$$

$$\{\phi_k(x)\} \to \phi(x).$$

where ϕ is a soln ~~to~~ the given initial value pbrm.

pbrm 1: consider the eqn $y' = \frac{y^3 e^x}{1+y^2} + x^2 \cos y$ on the strip. $S_a : |x| \leq a, (a > 0) : |y| < \infty$ show that f satisfies a lipschitz condition on the strip S_a and hence every initial value pbrm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln:

proof: Given $y' = \frac{y^3 e^x}{1+y^2} + x^2 \cos y$

$$\text{let } f(x,y) = \frac{y^3 e^x}{1+y^2} + x^2 \cos y$$

clearly $f(x,y)$ is continuous on the plane.

$$\frac{\partial f}{\partial y}(x,y) = \frac{[(1+y^2) 3y^2 - 2y \cdot y^3]}{(1+y^2)^2} e^x - x^2 \sin y$$

$$= \frac{3y^2 + 3y^4 - 2y^4}{(1+y^2)^2} e^x - x^2 \sin y$$

(5) $\frac{\partial f}{\partial y}(x,y) = \frac{3y^2+y^4}{(1+y^2)^2} e^x - x^2 \sin y$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{3y^2+y^4}{(1+y^2)^2} e^x + (-x^2 \sin y)\right|$$

$$\leq \left|\frac{3y^2(1+y^2/3)}{(1+y^2)^2} e^x\right| + |x^2|\,|\sin y|$$

$3 i \frac{y}{3}$

$2^2 = 4 \quad 1.3 < 4$

$\frac{2^2}{3} = 1.3$

$$\leq \left|\frac{3y^2(1+y^2)}{(1+y^2)^{\cancel{2}}}\right| e^{|x|} + |x|^2$$

$$= \left|\frac{3y^2}{y^2(1/y^2+1)}\right| e^{|x|} + |x|^2$$

$$< \frac{3}{1+|1/y^2|} e^{|x|} + |x|^2$$

$$\left|\frac{\partial f}{\partial x}(x,y)\right| < 3e^a + a^2$$

Hence f satisfies the lipschitz condition with lipschitz constant.

$$K = 3e^a + a^2$$

Also by corollary,

Suppose f is a real valued continuous function on the plane $|x| < \infty$, $|y| < \infty$ which satisfies a lipschitz condition on each strip

$$S_a : |x| \leq a,\ |y| < \infty$$

where a is a positive number. then every initial value pblm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln which exist for all real x.

Hence the initial value pblm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln.

(6) 2) consider the eqn $y' = (3x^2+1)\cos^2 y + (x^3-2x)\sin 2y$ on the strip. $S_a : |x| \le a$, $(a>0)$ $|y| < \infty$ show that f satisfies a lipschitz Condition on the strip S_a and hence every initial value pbm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln:

Soln:

Gn $y' = (3x^2+1)\cos^2 y + (x^3-2x)\sin 2y$

let $f(x,y) = (3x^2+1)\cos^2 y + (x^3-2x)\sin 2y$

clearly $f(x,y)$ is continuous on the plane.

$$\frac{\partial f}{\partial y}(x,y) = -(3x^2+1)2\cos y \sin y + (x^3-2x)2\cos 2y$$

$$\left|\frac{\partial f}{\partial y}(x,y)\right| = \left|-(3x^2+1)2\cos y \sin y + (x^3-2x)2\cos 2y\right|$$

$$\le |(3x^2+1)||\sin 2y| + |2(x^3-2x)||\cos 2y|$$

$$\le |3x^2+1| + |2x^3+(-4x)|$$

$$\le 3|x^2| + 1 + 2|x^3| + 4|x|$$

$$< 2a^3 + 3a^2 + 4a + 1$$

Hence f satisfies the lipschitz condition with lipchitz constant.

$$K = 2a^3 + 3a^2 + 4a + 1$$

Also by corollary

Hence the initial value pbm $y' = f(x,y)$,

$y(x_0) = y_0$ has soln.

3) Consider the eqn $y' = \frac{\cos y}{1-x^2}$ on the strip $S_a : |x| < a$, $(a>0)$, $|y| < \infty$ show that f satisfies a lipschitz condition on the strip S_a and hence every initial value pbm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln.

Soln: Gn $y' = \frac{\cos y}{1-x^2}$

let $f(x,y) = \frac{\cos y}{1-x^2}$

Clearly $f(x,y)$ is continuous on the plane.

(7) $\frac{\partial f}{\partial y}(x,y) = \frac{-\sin y}{1-x^2}$

$\left|\frac{\partial f}{\partial y}(x,y)\right| = \frac{|\sin y|}{|1-x^2|} \leq \frac{1}{|1-x^2|} < \frac{1}{1-a^2}$

Hence f satisfies the lipschitz condition with lipschitz constant.

$$k = \frac{1}{1-a^2}$$

Also by corollary,

Hence the initial value pbom $y' = f(x,y)$, $y(x_0) = y_0$ has a soln.

Approximation and Uniqueness solns.

Let us consider two initial value pbms

$y' = f(x,y)$, $y(x_0) = y_1 \rightarrow$ (1)

and $y' = g(x,y)$, $y(x_0) = y_2 \rightarrow$ (2)

where f, g are both continuous real valued functions on,

$$R : |x - x_0| \leq a, \ |y - y_0| \leq b \ (a, b > 0)$$

and (x_0, y_1), (x_0, y_2) are points.

Thm 5.2 Let f, g be continuous on R and suppose (7) f satisfies a lipschitz condition there with lipschitz Constant K. Let ϕ, ψ be solns of $y' = f(x,y)$, $y(x_0) = y_1 \rightarrow$ (1) and $y' = g(x,y)$, $y(x_0) = y_2 \rightarrow$ (2) respect on an interval I, containing x_0 with graph contained in R. Suppose there exists a non-negative constants ϵ, δ such that

(8) $|f(x,y) - g(x,y)| \leq \varepsilon$, (x,y) in R and
$|y_1 - y_2| < \delta$ then $|\phi(x) - \psi(x)| \leq \delta$

$$e^{K|x-x_0|} + \frac{\varepsilon}{K} e^{K|x-x_0|}.$$

Remark:

$K/K\uparrow$, if g is close to f and y_2 is close to y then the soln ψ of (2) on an interval I containing x_0 is close to a soln ϕ of (1) on I by the following consequence.

If we take $g = f$ and $y_0 = y_1 = y_2$ then $\varepsilon = 0$ and $\delta = 0$.

Hence proved.

corollary 1: uniqueness thrm.

Let f be a continuous on R satisfies a lipschitz condition on R. If ϕ and ψ are two solns of $y' = f(x,y)$, $y(x_0) = y_0$ on an interval I containing x_0 then $\phi(x) = \psi(x)$ $\forall$ x in I.

corollary 2:

Let f be continuous and satisfy a lipchitz condition on R. Let the g_K $(K = 1, 2 \ldots)$ be continuous on R. These are constants ε_K such that

$$|f(x,y) - g_K(x,y)| \leq \varepsilon_K \quad [(x,y) \text{ in } R]$$

for some constants $\varepsilon_K \to 0$ $(K \to \infty)$ and

let $y_K - y_0$ $(K \to \infty)$.

If ψ_K is a soln of $y' = g_K(x,y)$, $y(x_0) = y_K$ on an interval I containing x_0 & ϕ is a soln of $y' = f(x,y)$, $y(x_0) = y_0$ on I Then

$$\psi_K(x) \to \phi(x) \text{ on } I$$

Thm 5.2 proof:

from ① & ②,

$$\phi(x) = y_1 + \int_{x_0}^{x} f(t, \phi(t))\, dt$$

(9)

$$\psi(x) = y_2 + \int_{x_0}^{x} g(t, \psi(t))\, dt$$

Hence $\phi(x) - \psi(x) = y_1 - y_2 + \int_{x_0}^{x} [f(t, \phi(t)] - [g(t, \psi(t))]\, dt$

$$= y_1 - y_2 + \int_{x_0}^{x} [f(t, \phi(t)) - f(t, \psi(t)) + f(t, \psi(t)) - g(t, \psi(t))]\, dt$$

$$|\phi(x) - \psi(x)| \leq |y_1 - y_2| + \left|\int_{x_0}^{x} [f(t, \phi(t)) - f(t, \psi(t))]\, dt\right| + \left|\int_{x_0}^{x} [f(t, \psi(t)) - g(t, \psi(t))]\, dt\right|$$

$$\leq |y_1 - y_2| + \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \psi(t))|\, dt + \int_{x_0}^{x} |f(t, \psi(t)) - g(t, \psi(t))|\, dt$$

$$\leq \delta + k \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt + \int_{x_0}^{x} \varepsilon\, dt \text{ (by 9)}$$

where k is Lipschitz condition, for $x \geq x_0$.

$$|\phi(x) - \psi(x)| \leq \delta + k \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt + \varepsilon(x - x_0) \rightarrow (A)$$

If $E = \int_{x_0}^{x} |\phi(t) - \psi(t)|\, dt$

$$E' = |\phi(t) - \psi(t)|$$

∴ (A) becomes

$$E'(x) = \delta + k\, E(x) + \varepsilon(x - x_0)$$

$$E'(x) - k\, E(x) \leq \delta + \varepsilon(x - x_0)$$

Multiply by $e^{-k(x - x_0)}$, we get

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$$e^{-K(x-x_0)} E'(x) - K e^{-K(x-x_0)} E(x)$$

$$\leq \delta e^{-K(x-x_0)} + \varepsilon(x-x_0) e^{-K(x-x_0)}$$

$$= \left[E(x) e^{-K(x-x_0)}\right]' \leq \delta e^{-K(x-x_0)} + \varepsilon(x-x_0) e^{-K(x-x_0)}$$

Replace x by t,

$$\left[E(t) e^{-K(t-x_0)}\right]' \leq \delta e^{-K(t-x_0)} + \varepsilon(t-\varepsilon_0) e^{-K(t-x_0)}$$

$$E(x) e^{-K(x-x_0)} \leq \delta \left\{ \frac{e^{-K(x-x_0)}}{-K} - \frac{1}{-K} \right\} + \varepsilon \left[\frac{(t-x_0) e^{-K(t-x_0)}}{-K} \right]_{x_0}^{x}$$

$$+ \frac{\varepsilon}{K} \int_{x_0}^{x_0} e^{-K(t-x_0)} dt$$

$$\leq \frac{\delta}{K}\left[1 - e^{-K(x-x_0)}\right] - \frac{\varepsilon}{K}\left[(x-x_0) e^{-K(x-x_0)} + \frac{e^{-K(x-x_0)}}{K} - \frac{1}{K}\right]$$

$$\leq \frac{\delta}{K}\left[1 - e^{-K(x-x_0)}\right] + \frac{\varepsilon}{K^2}\left[-K(x-x_0) - 1\right] e^{-K(x-x_0)} + \frac{\varepsilon}{K^2}$$

multiply by $e^{K(x-x_0)}$.

$$E(x) \leq \frac{\delta}{K}\left[e^{K(x-x_0)} - 1\right] - \frac{\varepsilon}{K^2}\left\{K(x-x_0)+1\right\} + \frac{\varepsilon}{K^2} e^{K(x-x_0)}$$

$\therefore$ (A) becomes,

$$|\phi(x) - \psi(x)| \leq \delta + K\left\{ \frac{\delta}{K}\left(e^{K(x-x_0)} - 1\right) - \frac{\varepsilon}{K^2}\left\{K(x-x_0)+1\right\} + \frac{\varepsilon}{K^2} e^{K(x-x_0)} \right\} + \varepsilon(x-x_0)$$

$$\leq \delta + \delta e^{K(x-x_0)} - \delta - \varepsilon(x-x_0) - \frac{\varepsilon}{K} + \frac{\varepsilon}{K} e^{K(x-x_0)} + \varepsilon(x-x_0)$$

$$\leq \delta e^{K(x-x_0)} + \frac{\varepsilon}{K}\left\{e^{K(x-x_0)} - 1\right\} \text{ for } x \geq x_0$$

|||ly for $x \geq x_0$,

$$|\phi(x) - \psi(x)| \leq \delta e^{K(x_0-x)} + \frac{\varepsilon}{K}\left\{e^{K(x_0-x)} - 1\right\}$$

Hence $|\phi(x) - \psi(x)| \leq \delta e^{K|x-x_0|} + \frac{\varepsilon}{K}\left\{e^{K|x-x_0|} - 1\right\}$.

Existence and uniqueness of solns of system:

[Let f be a continuous vector valued function defined on,

(11) $R: |x-x_0| \leq a,\ |y-y_0| \leq b\ (a,b>0)$.

An initial value pbrm,

$$y' = f(x,y),\ y(x_0) = y_0 \rightarrow (1)$$

If ϕ is a soln of (1) on an interval I containing x_0. Such that $\phi(x_0) = y_0$

If $y_0 = (\alpha_1, \alpha_2, \ldots \alpha_n)$

The pbrm (1) can becomes,

$$y_1' = f(x, y_1, y_2, \ldots y_n)$$

$$y_2' = f(x, y_1, y_2, \ldots y_n)$$

$$\vdots$$

$$y_n' = f(x, y_1, y_2, \ldots y_n)$$

$$y_1(x_0) = \alpha_1,\ y_2(x_0) = \alpha_2, \ldots\ y_n(x_0) = \alpha_n.$$

NOTE: The successive approximation is defined by $\phi_0(x) = y_0$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt \quad [k=0,1,2..]\ ]$$

Example: Solve $y_1' = y_2,\ y_2' = -y_1,\ y(0) = (0,1)$.

Soln: Given $y_1' = y_2,\ y_2' = -y_1,\ y(0) = (0,1)$

$$f(x,y) = (x, y, y_2) = (y_2, -y_1)$$

and $\phi_0(x) = (0,1) = y_0,\ x_0 = 0$.

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt$$

$$= (0,1) + \int_0^x f(t, \phi_0(t))\, dt$$

$$= (0,1) + \int_0^x f(t,(0,1))\,dt$$

$$= (0,1) + \int_0^x (1,0)\,dt = (0,1) + [t,0]_0^x$$

$$= (0,1) + (x,0)$$

$$\phi_1(x) = (x,1)$$

$$\phi_2(x) = y_0 + \int_0^x f(t, \phi_1(t))\,dt$$

$$= (0,1) + \int_0^x f(t,(t,1))\,dt$$

$$= (0,1) + \int_0^x (1,-t)\,dt = (0,1) + \left[t, -\frac{t^2}{2}\right]_0^x$$

$$= (0,1) + (x, -x^2/2)$$

$$\phi_2(x) = (x, 1 - x^2/2)$$

$$\phi_3(x) = y_0 + \int_0^x f(t, \phi_2(t))\,dt$$

$$= (0,1) + \int_0^x f(t, t, 1 - t^2/2)\,dt$$

$$= (0,1) + \int_0^x (1 - t^2/2, -t)\,dt$$

$$= (0,1) + \left[\left(t - t^3/6, \frac{-t^2}{2}\right)\right]_0^x$$

$$= (0,1) + \left[x - \frac{x^3}{3!}, -\frac{x^2}{2}\right]$$

$$= \left(x - \frac{x^3}{3!}, 1 - \frac{x^2}{2!}\right)$$

which ~~shows~~ shows that all ϕ_k exists for all real x and $\phi_k(x) \to \phi(x)$.

where $\phi(x) = (\sin x, \cos x)$ is a soln of the gn pbm.

Thrm 5.3 Local existence:

If f be a continuous vector valued function defined on $R: |x-x_0| \leq a$, $|y-y_0| \leq b$ $(a, b > 0)$ and suppose f satisfies a lipschitz condition on R. If M is a constant such that $|f(x,y)| \leq M$ $\forall (x,y)$ in R.

The successive approximation $\{\phi_k\}$ $(k = 0,1,2 \ldots)$ given by,

$$\phi_0(x) = y_0$$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt \quad (k = 0,1,2 \ldots)$$

converge on the interval

$I: |x-x_0| \leq \alpha =$ minimum $(a, b/M)$ to a soln ϕ of the initial value pbrm,

with existence and uni $y' = f(x,y)$, $y(x_0) = y_0$ on I.

Thrm 5.4

If f satisfies the same conditions as in Thrm 5.3 and K is a lipschitz constant for f in R Then $|\phi(x) - \phi_k(x)| \leq \frac{M}{K} \frac{(K\alpha)^{k+1}}{(k+1)!} e^{K\alpha}$ for all x in I.

Thrm 5.5 Non-Local existence.

Let f be a continuous vector valued function defined on,

$S: |x-x_0| \leq a$, $|y| \leq \infty$ $(a > 0)$

and satisfy these a lipschitz condition. then the successive approximation $\{\phi_k\}$ for the pbrm.

$y' = f(x,y)$, $y(x_0) = y_0$ $(|y_0| < \infty)$ exist on $|x-x_0| \leq a$ and converge these to a soln ϕ of this pbrm.

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Corollary: Suppose f is a continuous vector valued function defined on $|x|<\infty$, $|y|<\infty$ and satisfies a Lipschitz condition on each strip $|x| \le \infty$, $|y| \le \infty$

where a is any positive number.

R. Then every initial value pbrm $y' = f(x,y)$, $y(x_0) = y_0$ has a soln which exists for all real x.

Thrm 5.6 Approximation and uniqueness.

Let f, g be two continuous vector valued fns and defined on $R: |x-x_0| \le a$, $|y-y_0| \le b$ $(a, b > 0)$ and Suppose f satisfies a Lipschitz condition on R with Lipschitz constant K. Suppose ϕ, ψ are Solns of the pbrm,

$$y' = f(x,y), \quad y(x_0) = y_1$$

$$y' = g(x,y), \quad y(x_0) = y_2$$

respectively on some interval I containing x_0,

If for $\varepsilon, \delta \ge 0$.

$|f(x,y) - g(x,y)| \le \varepsilon$ (all (x,y) in R)

and $|y_1 - y_2| \le \delta$ then

$$|\phi(x) - \psi(x)| \le \delta e^{K(x-x_0)} + \frac{\varepsilon}{K}\left(e^{K(x-x_0)} - 1\right)$$

for all x in I.

In particular the pbrm

$$y' = f(x,y), \quad y(x_0) = y_0$$

has atmost one Soln on any interval I containing x_0.